Question about one of the geodesic equations for the Schwarzschild metric

I have no idea if this is an "A" level question, but I will put that down.

From the Schwarzschild metric, and with the help of the Maxima program, one of the geodesic equations is:

$$0 = \frac{d^2 t}{ds^2} + \frac{r_s}{r\left(r - r_s\right)} \frac{dr}{ds} \frac{dt}{ds} .$$

I believe this integrates to the following, with $\left(g \equiv 1 - r_s / r\right)$:

$$\begin{aligned} \frac{dt}{ds} &= k \frac{r}{\left(r - r_s\right)} \\ &= \frac{k}{g} \end{aligned}$$

(1) Is that correct so far?

(2) But when I "solve" for $\left(dt/ds\right)$ with the metric, I don't get the same thing. Shouldn't I get something "similar" or recognizable when all the other differentials are zero?

$$\begin{aligned} \left(ds\right)^2 &= g\left(dt\right)^2 - \frac{1}{g}\left(dr\right)^2 - \left(r d\theta\right)^2 - \left(r \sin\theta d\phi\right)^2 \\ 1 &= g\left(\frac{dt}{ds}\right)^2 - \left(0\right)^2 - \left(0\right)^2 - \left(0\right)^2 \\ \frac{dt}{ds} &= \frac{1}{\sqrt{g}} \end{aligned}$$

Obviously g is not equal to the square root of g, so I must be missing something. Any help is appreciated.