CHAPTER 12

IDENTICAL PARTICLES

In any discussion of multielectron atoms, molecules, solids, nuclei, or elementary particles, we face systems that involve identical particles. As we will discuss in this chapter, the truly indistinguishable nature of identical particles within quantum mechanics has profound consequences for the way the physical world behaves.

12.1 INDISTINGUISHABLE PARTICLES IN QUANTUM MECHANICS

As far as we can tell, all electrons are identical. They all have the same mass, the same charge, and the same intrinsic spin. There are no additional properties, such as color, that allow us to distinguish one electron from another. Yet within classical mechanics, identical particles are, in principle, distinguishable. You don't have to paint one of them red and one of them green to be able to tell two identical particles apart. If at some initial time you specify the positions and the velocities $(\mathbf{r}_1, \mathbf{v}_1)$ and $(\mathbf{r}_2, \mathbf{v}_2)$ of two interacting particles, you can calculate their positions and velocities at all later times. The particles follow well-defined trajectories, so you don't need to actually observe the particles to be sure which is which when you find one of the particles at a later time. In any case, within classical theory you would, in principle, be permitted to make measurements of the particles' positions and velocities without influencing their motions so that you could actually follow the trajectories of the two particles and thus keep track of them.

Life in the real world is different, at least on the microscopic level. As we have seen in Chapter 8, in many microscopic situations there is no well-defined trajectory that a particle follows. The particle has amplitudes to take all paths. Or in the language of wave functions, each of the particles may have an amplitude to be at a variety of overlapping positions, as indicated in Fig. 12.1, so we cannot be sure

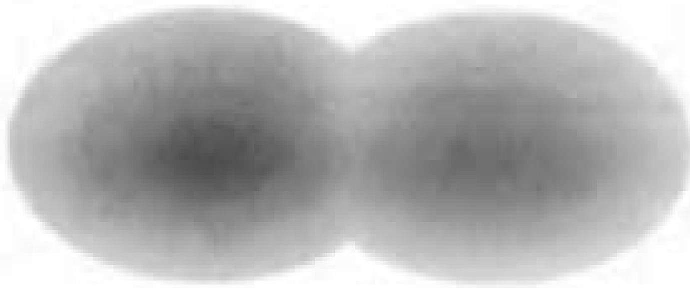

FIGURE 12.1
A schematic diagram indicating the position probability distribution for two particles. Since these distributions overlap, there is no way to be sure which particle we have detected if we make a measurement of the particle's position and the two particles are identical.

which of the particles we have found if we make a subsequent measurement of the particle's position. Moreover, any attempt to keep track of the particle by measuring its position is bound to change fundamentally the particle's quantum state.

With these considerations in mind, let's see what types of states are allowed for a pair of identical particles. We specify a two-particle state by

$$|a, b\rangle = |a\rangle_1 \otimes |b\rangle_2 \tag{12.1}$$

where a single-particle state such as $|a\rangle_1$ specifies the state of particle 1 and $|b\rangle_2$ specifies the state of particle 2.

We introduce the exchange operator $\hat{P}_{12}$, which is defined by

$$\hat{P}_{12}|a, b\rangle = |b, a\rangle \tag{12.2a}$$

or

$$\hat{P}_{12}(|a\rangle_1 \otimes |b\rangle_2) = |b\rangle_1 \otimes |a\rangle_2 \tag{12.2b}$$

As an example, the effect of the exchange operator on the state $|\mathbf{r}_1, +\mathbf{z}\rangle_1 \otimes |\mathbf{r}_2, -\mathbf{z}\rangle_2$, which has particle 1 at position $\mathbf{r}_1$ with $S_z = \hbar/2$ and particle 2 at position $\mathbf{r}_2$ with $S_z = -\hbar/2$, is to produce the state $|\mathbf{r}_2, -\mathbf{z}\rangle_1 \otimes |\mathbf{r}_1, +\mathbf{z}\rangle_2$, which has particle 1 at position $\mathbf{r}_2$ with $S_z = -\hbar/2$ and particle 2 at position $\mathbf{r}_1$ with $S_z = \hbar/2$ (see Fig. 12.2). The exchange operator interchanges the particles, switching the subscript labels 1 and 2 on the states. Since for any physical state of two identical particles we cannot tell if we have exchanged the particles, the "exchanged" state must be the same physical state and therefore can differ from the initial state by at most an overall phase:

$$\hat{P}_{12}|\psi\rangle = e^{i\delta}|\psi\rangle = \lambda|\psi\rangle \tag{12.3}$$

Thus the allowed physical states are eigenstates of the exchange operator with eigenvalue λ. Applying the exchange operator twice yields the identity operator. Therefore

$$\hat{P}_{12}^2|\psi\rangle = \lambda^2|\psi\rangle = |\psi\rangle \tag{12.4}$$

which shows that $\lambda^2 = 1$, or $\lambda = \pm 1$ are the two allowed eigenvalues.[1]

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[1] There are exceptions to this rule in two-dimensional systems. See the article "Anyons" by F. Wilczek, *Scientific American*, May 1991, p. 58.

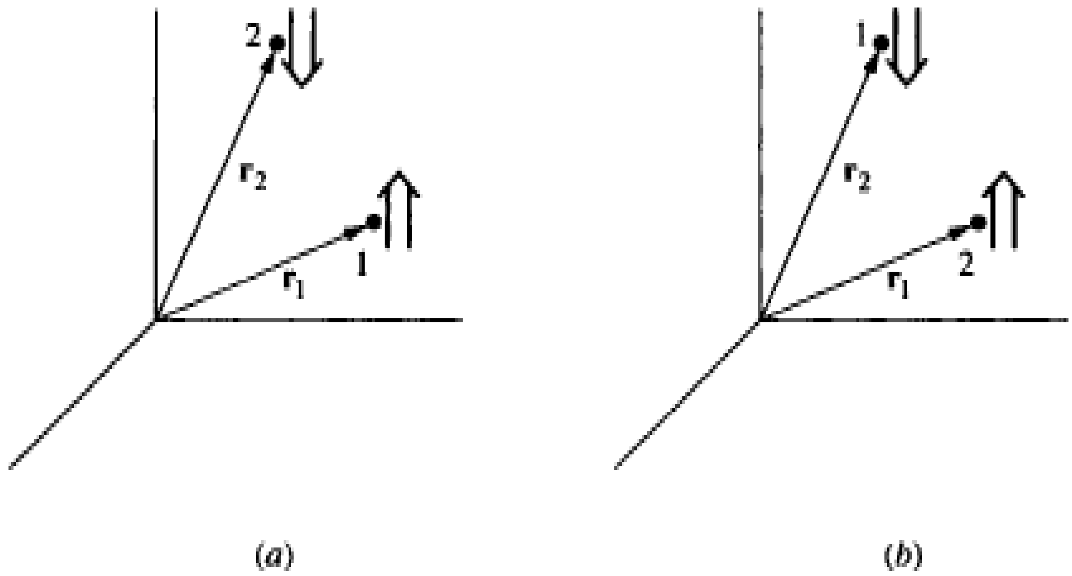

FIGURE 12.2
The effect of the exchange operator on a state of two spin-$\frac{1}{2}$ particles as shown in (*a*) is to exchange both the positions and the spins (indicated by the double arrow), as shown in (*b*).

Clearly, if the two identical particles are each in the same state $|a\rangle$, they are in an eigenstate of the exchange operator with eigenvalue $\lambda = 1$:

$$\hat{P}_{12}|a, a\rangle = |a, a\rangle \tag{12.5}$$

indicating that the state is symmetric under exchange. If $b \neq a$, we can find the linear combinations of the two states $|a, b\rangle$ and $|b, a\rangle$ that are eigenstates of the exchange operator. The matrix representation of the exchange operator using these states as a basis is given by

$$\hat{P}_{12} \rightarrow \begin{pmatrix} \langle a, b|\hat{P}_{12}|a, b\rangle & \langle a, b|\hat{P}_{12}|b, a\rangle \\ \langle b, a|\hat{P}_{12}|a, b\rangle & \langle b, a|\hat{P}_{12}|b, a\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{12.6}$$

where we have used action of the exchange operator as given in (12.2) and assumed that the two states $|a, b\rangle$ and $|b, a\rangle$ are normalizable and orthogonal. Thus the condition that the eigenvalue equation (12.3) has a nontrivial solution is given by

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \tag{12.7}$$

which also yields $\lambda = \pm 1$ as before. Substituting the eigenvalues into the eigenvalue equation, we find that the eigenstates corresponding to these eigenvalues are given by

$$|\psi_S\rangle = \frac{1}{\sqrt{2}}|a, b\rangle + \frac{1}{\sqrt{2}}|b, a\rangle \qquad \lambda = 1 \tag{12.8a}$$

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}|a, b\rangle - \frac{1}{\sqrt{2}}|b, a\rangle \qquad \lambda = -1 \tag{12.8b}$$

where the subscripts S and A indicate that these two eigenstates are symmetric and antisymmetric, respectively, under the interchange of the two particles. Notice that two identical particles must be in *either* the state $|\psi_S\rangle$ *or* the state $|\psi_A\rangle$, but they cannot be in a superposition of these states, for then exchanging the two particles does not lead to a state that differs from the initial state by an overall phase:

$$\hat{P}_{12}\big(c_S|\psi_S\rangle + c_A|\psi_A\rangle\big) = c_S|\psi_S\rangle - c_A|\psi_A\rangle \tag{12.9}$$

Thus the particles must make a choice between $|\psi_S\rangle$ and $|\psi_A\rangle$. In fact, it turns out that Nature makes the choice for them in a strikingly comprehensive way:

Particles with an integral intrinsic spin, $s = 0, 1, 2, \ldots$, are found to be only in symmetric states and are called *bosons*; these particles obey Bose-Einstein statistics.[2] Examples of such particles include fundamental elementary particles such as photons, gluons, the $W_{\pm}$ and Z_0 intermediate vector bosons, and the graviton—particles that mediate the electromagnetic, strong, weak, and gravitational interactions, respectively—as well as composite particles such as pions and nuclei such as He^4.

Particles with half-integral intrinsic spin, $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, are found to be only in antisymmetric states and are called *fermions*; these particles obey Fermi-Dirac statistics. Examples of such particles include fundamental elementary particles such as electrons, muons, neutrinos, and quarks, as well as composite particles such as protons, neutrons, and nuclei such as He^3.

At the level of nonrelativistic quantum mechanics, this relationship between the intrinsic spin of the particle and the exchange symmetry of the quantum state is a law of nature—often referred to as the spin-statistics theorem—that we must accept as a given. We can take comfort in the fact that this spin-statistics theorem can be shown to be a necessary consequence of relativistic quantum field theory.[3] In Chapter 14 we consider the fully relativistic quantum field theory for photons, and we can then see why, as an example, photons must indeed be bosons.

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[2] The symmetry requirement on the allowed quantum states of identical bosons leads to a statistical distribution function for an ensemble of N identical bosons in thermal equilibrium at a temperature T that is different from the classical Boltzmann distribution function. In particular, the number of bosons in a particular quantum state with energy E is given by

$$n(E) = \frac{1}{e^{\alpha}e^{E/kT} - 1}$$

where the value of α is chosen so as to ensure that the total number of particles is indeed N. On the other hand, the antisymmetry requirement on the allowed quantum states for an ensemble of N identical fermions leads to the distribution function

$$n(E) = \frac{1}{e^{\alpha}e^{E/kT} + 1}$$

Note: $n(E)$ can be very large for the Bose-Einstein distribution, while for the Fermi-Dirac distribution $n(E) \leq 1$. For a derivation of these quantum distribution functions, see, for example, F. Reif, *Fundamentals of Statistical and Thermal Physics*, McGraw-Hill, New York, 1965, Chapter 9.

[3] A comprehensive but advanced discussion is given by R. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That*, W. A. Benjamin, New York, 1964.