1.5-2 Consider the incompressible Newtonian fluid flowing, under the pressure gradient dp/dz, between two stationary horizontal plates separated by a gap of $2L$, as shown in Fig. P1.5-2 (see p. 108). The volume flow rate Q is known. Find the steady-state velocity distribution and the shear force acting on the two plates by the fluid.

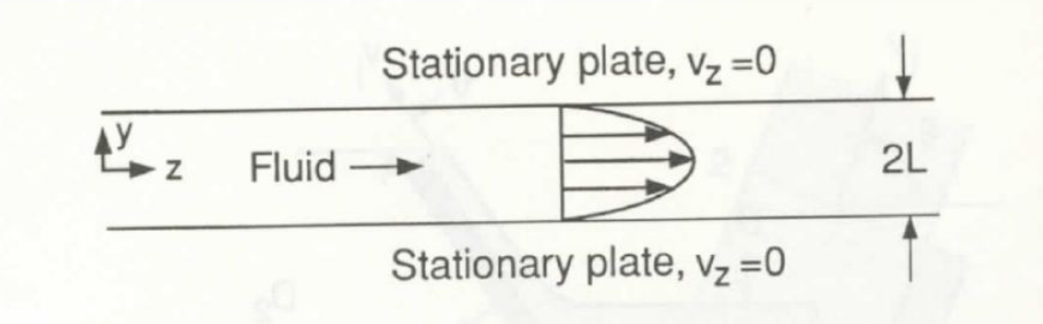

Fig. P1.5-2

Equation of continuity in rect. coord.:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y + \frac{\partial}{\partial z}\rho v_z = 0$$

After simplification:

$$\frac{\partial}{\partial y} v_y = 0$$

Now the equation of motion in rect coord. in the z direction:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial y} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

Simplifications:

$$\mu\frac{d^2 v_z}{dy^2} = \frac{dP}{dz}$$

If my understanding is correct, since neither side of the ODE is dependent on the same variable; they must be constants. Thus;

$$\mu\,\frac{d^2 v_z}{dy^2} = \frac{P_L - P_{-L}}{L-(-L)} = \frac{P_L - P_{-L}}{2L}$$

$$\mu\,\frac{d^2 v_z}{dy^2} = \frac{P_L - P_{-L}}{2L}$$

Note that the "P_{-L}" is P subscript "-L"

Rearrange:

$$\frac{d^2 v_z}{dy^2} = \frac{P_L - P_{-L}}{2\mu L}$$

Integrate once:

$\frac{dv_z}{dy} = \frac{P_L - P_{-L}}{2 \mu L} \, y + c_1$

Again:

$v_z = \frac{P_L - P_{-L}}{4 \mu L} \, y^2 + c_1 \, y + c_2$

I choose the boundary conditions:
$v_z(L) = 0$
$v_z(-L) = 0$

I found that:
$c_1 = 0$
$c_2 = -\frac{P_L - P_{-L}}{4 \mu} \, L$

So;

$\boldsymbol{v_z = \frac{y^2 - L^2}{4 \mu L} \, (P_L - P_{-L})}$

As for the Shear stress at L and -L;

Newton’s law of viscosity;
$\tau_{yz} = -\mu \, \frac{dv_z}{dy}$

$\tau_{yz} = -\mu \, \frac{d}{dy} \left(\frac{y^2 - L^2}{4 \mu L} \, (P_L - P_{-L}) \right)$

$\tau_{yz} = -\frac{y}{2 L} \, (P_L - P_{-L})$

$\boldsymbol{\tau_{yz}\big|_L = -\frac{1}{2} \, (P_L - P_{-L})}$

$\boldsymbol{\tau_{yz}\big|_{-L} = \frac{1}{2} \, (P_L - P_{-L})}$