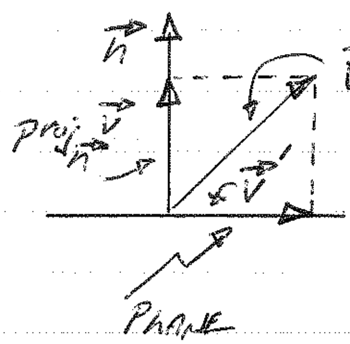
TERM TEST SOLUTIONS

Q1: NORMAL TO PLANE $\vec{n} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$



$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\vec{v}$ IS ASSOCIATED WITH POINT (x, y, z)

WANT TO FIND $\vec{v}'$

$$\vec{v}' = \vec{v} - proj_{\vec{n}} \vec{v}$$

$$proj_{\vec{n}} \vec{v} = \frac{\vec{v} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n} = \frac{7x - y + 3z}{59} \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$$

$$\vec{v}' = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{7x - y + 3z}{59} \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{1}{59} \begin{bmatrix} 49x - 7y + 21z \\ -7x + y - 3z \\ 21x - 3y + 9z \end{bmatrix}$$

$$= \frac{1}{59} \begin{bmatrix} 59x - 49x + 7y - 21z \\ 59y + 7x - y + 3z \\ 59z - 21x + 3y - 9z \end{bmatrix}$$

$$= \frac{1}{59} \begin{bmatrix} 10x + 7y - 21z \\ 7x + 58y + 3z \\ -21x + 3y + 50z \end{bmatrix}$$

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$$\vec{v}\,' = \frac{1}{59}\begin{bmatrix} 10 & 7 & -21 \\ 7 & 58 & 3 \\ -21 & 3 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore$ MATRIX THAT TRANSFORMS $\vec{v}$ TO $\vec{v}\,'$ IS:

$$\frac{1}{59}\begin{bmatrix} 10 & 7 & -21 \\ 7 & 58 & 3 \\ -21 & 3 & 50 \end{bmatrix}$$

Q2:

a) $AX=B$ HAS TWO SOLUTIONS, X_1 AND X_2.

$\therefore$ $AX_1=B$ AND $AX_2=B$

IS $tX_1+(1-t)X_2$ ALSO A SOLUTION?

$$A(tX_1+(1-t)X_2)$$
$$=tAX_1+(1-t)AX_2$$
$$=tB+(1-t)B$$
$$=B$$

$\therefore$ $tX_1+(1-t)X_2$ IS ALSO A SOLUTION.

b) IF A SYSTEM HAS TWO SOLUTIONS, THEN $tX_1+(1-t)X_2$ IS ALSO A SOLUTION. SINCE t CAN TAKE ON ANY SCALAR VALUE, THEN THERE ARE INFINITELY MANY SOLUTIONS DESCRIBED BY $tX_1+(1-t)X_2$.

NOTE THAT: $t=1$ $X=X_1$
$t=0$ $X=X_2$

c) SET OF SOLUTIONS CONTAINS:

$$X = tX_1 + (1-t)X_2$$
$$= X_2 + t(X_1 - X_2)$$

COMPARE TO VECTOR EQUATION OF A LINE:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t\vec{d}$$

$\therefore$ THE SET OF SOLUTIONS CONTAINS A LINE:

$$X = X_2 + t(X_1 - X_2)$$

WHERE X_2 IS LIKE A POINT ON THE LINE AND $X_1 - X_2$ IS LIKE A DIRECTION VECTOR ASSOCIATED WITH THE LINE.

Q3: NEED TO EXAMINE THE SOLUTIONS TO:

$$c_1\begin{bmatrix}3\\2\\7\end{bmatrix}+c_2\begin{bmatrix}4\\1\\-3\end{bmatrix}+c_3\begin{bmatrix}6\\-1\\-23\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

PUT IN $AX=B$ FORM:

$$\begin{bmatrix}3&4&6\\2&1&-1\\7&-3&-23\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

AUGMENTED MATRIX FORM:

$$\left[\begin{array}{ccc|c}3&4&6&0\\2&1&-1&0\\7&-3&-23&0\end{array}\right]$$

$$\Downarrow \text{RNF}$$

$$\left[\begin{array}{ccc|c}1&0&-2&0\\0&1&3&0\\0&0&0&0\end{array}\right]$$

LEADING VARIABLES: c_1, c_2
FREE VARIABLE: c_3

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=c_3\begin{bmatrix}2\\-3\\1\end{bmatrix}$$

INFINITELY MANY SOLUTIONS.

$\therefore$ VECTORS ARE LINEARLY DEPENDENT.

Q4: IS THERE A SOLUTION TO THIS PROBLEM:

$$\begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix} \vec{v}$$

LET $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$$

AUGMENTED MATRIX FORM:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{array}\right]$$

$$\Downarrow \text{RNF}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\therefore \vec{v} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$$

Q5:

a) LOOKING FOR SOLUTIONS TO:

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

NEED TO FIND ROOTS:

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4ad + 4bc}}{2}$$

$$= \frac{a+d}{2} \pm \sqrt{\frac{a^2 + 2ad + d^2 - 4ad + 4bc}{4}}$$

$$= \frac{a+d}{2} \pm \sqrt{\left(\frac{a-d}{2}\right)^2 + bc}$$

b) TO HAVE TWO DISTINCT REAL EIGENVALUES:

$$\left(\frac{a-d}{2}\right)^2 + bc > 0$$

OR

$$(a-d)^2 > -4bc$$

c) IF $(a-d)^2 = -4bc$, EIGENVALUES BECOME EQUAL:

$$\lambda = \frac{a+d}{2}, \frac{a+d}{2}$$

FIND EIGENVECTORS:

$$A\vec{u} = \lambda\vec{u}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{a+d}{2}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$au_1 + bu_2 = \frac{a+d}{2}u_1$$

$$cu_1 + du_2 = \frac{a+d}{2}u_2$$

SIMPLIFYING:

$$(a-d)u_1 + 2bu_2 = 0$$
$$2cu_1 - (a-d)u_2 = 0$$

AUGMENTED MATRIX:

$$\left[\begin{array}{cc|c} a-d & 2b & 0 \\ 2c & -(a-d) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{2b}{a-d} & 0 \\ 2c & -(a-d) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{2b}{a-d} & 0 \\ 0 & -(a-d) - \frac{4bc}{a-d} & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{2b}{a-d} & 0 \\ 0 & \frac{-(a-d)^2 - 4bc}{a-d} & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{2b}{a-d} & 0 \\ 0 & 0 & 0 \end{array}\right]$$ BECAUSE $(a-d)^2 = -4bc$

LEADING: u_1
FREE: u_2

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_2 \begin{bmatrix} \frac{-2b}{a-d} \\ 1 \end{bmatrix}$$

96: a) LOOKING FOR VALUES OF x, y, z THAT ARE COMMON TO BOTH LINES:

$$\begin{aligned} 4+3t &= 13+4s \\ 7+t &= 10+2s \\ 2 &= 2 \end{aligned}$$

LOOK FOR SOLUTIONS TO s AND t:

$$\begin{aligned} 3t-4s &= 9 \\ t-2s &= 3 \end{aligned}$$

AUGMENTED MATRIX:

$$\left[\begin{array}{cc|c} 3 & -4 & 9 \\ 1 & -2 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -4/3 & 3 \\ 1 & -2 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -4/3 & 3 \\ 0 & -2/3 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -4/3 & 3 \\ 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0 \end{array}\right]$$

NO FREE VARIABLES

LEADING VARIABLES: s, t

$$t = 3$$
$$s = 0$$

POINT IN COMMON IS $(13, 10, Z)$

b) THE SAME IS NOT EXPECTED IN R^3. ANY TWO NONPARALLEL LINES IN R^2 WILL INTERSECT BUT NONPARALLEL LINES IN R^3 DO NOT NECESSARILY INTERSECT.

c) A RANDOM LINE AND RANDOM PLANE WILL LIKELY INTERSECT IN R^3 BECAUSE AS LONG AS THE LINE IS NOT PARALLEL TO THE PLANE THEY WILL HAVE A POINT IN COMMON.