Ben Orin, benorin@umail.ucsb.edu
Excerpt from Dirichlet Integrals Paper

Note: $C^n=[0,1]^n$ is the unit hypercube and $n\in\mathbb{Z}^+$ throughout.

Definition: The generalized hypergeometric function of a single variable

$$ {}_pF_q\left(\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix};z\right):=\sum_{\lambda=0}^{\infty}\left\{\frac{z^\lambda}{\lambda!}\prod_{k=1}^{p}\left[\frac{\Gamma(a_k+\lambda)}{\Gamma(a_k)}\right]\prod_{j=1}^{q}\left[\frac{\Gamma(b_j)}{\Gamma(b_j+\lambda)}\right]\right\} $$

Theorem:

$$ {}_{n+1}F_n\left(\begin{matrix}a_1,a_2,\ldots,a_n,a_{n+1}\\ b_1,b_2,\ldots,b_n\end{matrix};z\right)=\prod_{k=1}^{n}\left[\frac{\Gamma(b_k)}{\Gamma(a_{k+1})\Gamma(b_k-a_{k+1})}\right]\iint_{C^n}\cdots\int\prod_{q=1}^{n}\left[t_q^{a_{q+1}-1}(1-t_q)^{b_q-a_{q+1}-1}\right]\left(1-z\prod_{\lambda=1}^{n}t_\lambda\right)^{-a_1}dt_1\cdots dt_{n-1}dt_n $$

$$ {}_nF_n\left(\begin{matrix}a_1,a_2,\ldots,a_n\\ b_1,b_2,\ldots,b_n\end{matrix};z\right)=\prod_{k=1}^{n}\left[\frac{\Gamma(b_k)}{\Gamma(a_k)\Gamma(b_k-a_k)}\right]\iint_{C^n}\cdots\int\exp\left(z\prod_{\lambda=1}^{n}t_\lambda\right)\prod_{q=1}^{n}\left[t_q^{a_q-1}(1-t_q)^{b_q-a_q-1}\right]dt_1\cdots dt_{n-1}dt_n $$

Corollary:

Lerch Transcendent: $\Phi(z,n,y):=\sum_{q=0}^{\infty}\frac{z^q}{(q+y)^n}=\iint_{C^n}\cdots\int\prod_{k=1}^{n}\left(\lambda_k^{y-1}\right)\left(1-z\prod_{q=1}^{n}\lambda_q\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Legendre Chi Function: $\chi_n(z):=\sum_{q=0}^{\infty}\frac{z^{2q+1}}{(2q+1)^n}=z\iint_{C^n}\cdots\int\left(1-z^2\prod_{q=1}^{n}\lambda_q^2\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Polygamma Function: $\psi_n(z)=\sum_{q=0}^{\infty}\frac{(-1)^{n+1}n!}{(z+q)^{n+1}}=(-1)^{n+1}n!\iint_{C^{n+1}}\cdots\int\prod_{k=1}^{n+1}\left(\lambda_k^{z-1}\right)\left(1-\prod_{q=1}^{n+1}\lambda_q\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Polylogarithm of Order n: $\mathrm{Li}_n(z):=\sum_{q=1}^{\infty}\frac{z^q}{q^n}=z\iint_{C^n}\cdots\int\left(1-z\prod_{q=1}^{n}\lambda_q\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Hurwitz Zeta Function: $\zeta(n,y):=\sum_{q=0}^{\infty}\frac{1}{(q+y)^n}=\iint_{C^n}\cdots\int\prod_{k=1}^{n}\left(\lambda_k^{y-1}\right)\left(1-\prod_{q=1}^{n}\lambda_q\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Riemann Zeta Function: $\zeta(n):=\sum_{q=1}^{\infty}\frac{1}{q^n}=\iint_{C^n}\cdots\int\left(1-\prod_{q=1}^{n}\lambda_q\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Dirichlet Beta Function: $\beta(n):=\sum_{q=0}^{\infty}\frac{(-1)^q}{(2q+1)^n}=\iint_{C^n}\cdots\int\left(1+\prod_{q=1}^{n}\lambda_q^2\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Dirichlet Eta Function: $\eta(n):=\sum_{q=1}^{\infty}\frac{(-1)^{q-1}}{q^n}=\iint_{C^n}\cdots\int\left(1+\prod_{q=1}^{n}\lambda_q\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$

Dirichlet Lambda Function: $\lambda(n):=\sum_{q=0}^{\infty}\frac{1}{(2q+1)^n}=\iint_{C^n}\cdots\int\left(1-\prod_{q=1}^{n}\lambda_q^2\right)^{-1}d\lambda_1\cdots d\lambda_{n-1}d\lambda_n$