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VIBRATIONAL MODES OF THICK CYLINDERS OF FINITE LENGTH

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The natural frequencies and mode shapes of finite length thick cylinders are of considerable engineering importance. A comprehensive classification of the modes of such thick cylinders, based on three-dimensional mode shapes, is presented in this paper. In addition to the 5 groups consisting of pure radial, radial motion with radial shearing, extensional, axial bending, and global modes, as previously adopted for thin cylinders, a further sixth circumferential category is proposed. This classification, together with the numbers of both the circumferential and the longitudinal nodes, is sufficient to identify each mode of a finite length thick cylinder. The classification for the modes of thick cylinders is applied to four groups of cylinders whose radial thickness to median radius ratio varies from 0·1 to 0·4. Each group contains a set of 8 cylinders with similar inside and outside diameters, but with different axial lengths; these were used to verify the validity of the classifications, and to study the effects of varying axial length and varying radial thickness on each of the different types of modes. Analytical finite element analysis was applied to all four groups, and experimental analysis was applied to those cylinders whose radial thickness to the medium radius ratio was 0·4. The results support the method of classification, although very short cylinders behave essentially as annular circular plates. The effects of varying axial length and radial thickness on the vibrational modes are such that all modes can be broadly categorized as either pure radial modes, or non-pure radial modes. The natural frequencies of the former are dependent upon only the radial dimensions of the models, while the natural frequencies of the latter are dependent upon both axial length and radial thickness.

1. INTRODUCTION

The vibrations of hollow cylinders are of considerable engineering importance as such cylinders have numerous applications in practice. A number of papers for the prediction of the resonant frequencies of cylinders have been published over the years. Early investigations of the vibrations of hollow cylinders based on the linear three-dimensional theory of elasticity were by Greenspon [1] and Gazis [2]. McNiven *et al*. [3] developed a three-mode theory, i.e. the longitudinal, first radial and first axial shear modes, for the axisymmetric vibrations of hollow cylinders. McNiven and Shah [4] further extended this to investigate the end mode. Tabulated data of the natural frequencies and mode shapes of hollow and solid circular cylinders, with infinite length, were published by Armenakas *et al*. [5]. The general problem of three-dimensional vibrations of a finite length circular cylinder, with traction free surfaces, was first studied by Gladwell and Vijay [6]. Hutchinson and El-Azhari [7] found the natural

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frequencies for the vibrations of hollow elastic circular cylinders, with traction free surfaces, by using a series solution. Singal and Williams [8] investigated the vibrations of thick circular cylinders by using the energy method based on the three-dimensional theory of elasticity. Also investigations of thick circular cylinders can be found in studies of cylindrical shells, a summary and comparison of which can be found in reference [9].

In so far as the experimental investigation is concerned, McMahon [10] gave experimental results for the natural frequencies of vibrations of solid, isotropic, elastic cylinders with free boundaries. Singal and Williams [8] gave experimental results for both the natural frequencies and mode shapes of a series of thick circular cylinders with traction free surfaces.

The vibrational analysis of thick cylinders of finite length is more complicated than that of thin cylinders. For the case of traction free surfaces, there are no closed form solutions [6]. Either an analytical approach using series solutions, or a numerical approach using the finite element method is often used. In the past, some papers, such as reference [6], were restricted to discussing natural frequencies, and the relationship between the natural frequency and the dimensions of the thick cylinder. In references [8, 11] the authors discussed both the natural frequencies and mode shapes of thick cylinders. The mode shapes were described in the circumferential and longitudinal directions separately. Such a method of description may be satisfactory for the mode shapes of thin cylinders, but it is not complete enough to accurately describe the mode shapes of thick cylinders. References [8, 11] further indicated that several combinations of the same circumferential and longitudinal node numbers existed in the frequency range investigated. In their theoretical analysis, Girgis and Verma [12] indicated that there were three such frequencies for each combination of circumferential and longitudinal node numbers. The first frequency was predicted as being a predominantly radial vibration, the second as being predominantly axial, and finally, the third and highest as being predominantly tangential in natural. Such an explanation is still not sufficiently exact because, as will be shown, some of the mode shapes of thick cylinders are not readily represented by these descriptors.

In this paper, a classification of the vibrational modes of thick cylinders of finite length, based on three-dimensional mode shapes, is proposed. For such a classification, information about both the natural frequencies and mode shapes is necessary. The finite element method is used in this paper to calculate these. Using commercially available finite element programs, it is relatively easy to obtain the necessary information upon which the mode classification can be based. The classification is then applied in a study of how variations of both cylinder axial length and radial thickness influence the natural frequencies of a cylinder.

According to the classification of the modes of thick cylinders used by Singal and Williams [8], the natural frequencies and the mode shapes of different kinds of modes vary with changing length in a number of different ways. In this paper, the variation in the natural frequencies and the node shapes of several sets of thick cylinders, with varying axial length and radial thickness, are analysed theoretically using a finite element technique. The natural frequencies and mode shapes of a series of cylinder models, whose radial thickness to median radius ratio was 0·4, were verified by experiment. The results of the analytical and experimental investigations are presented here to further verify the validity of the classification, and to reveal the relationships which exist between the different categories of modes and the two variables: axial length and radial thickness.

2. NATURAL FREQUENCY AND MODE SHAPE

The equation of motion of a unit infinitesimal element can be expressed by the displacement tensor as

$$\mu U_{i,mm} + (\lambda + \mu) U_{m,mi} + F_i = \rho U_{i,tt}, \tag{1}$$

where U is the displacement tensor, F is the applied force, λ is the Lame-coefficient, μ is the shear modulus of elasticity, ρ is the material density, and $i, m = x, y, z$. The corresponding finite element equation of motion for free vibration is

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = 0, \tag{2}$$

where

$$\mathbf{M} = \int_\Omega \rho W U_{,i} \, \mathrm{d}\Omega, \qquad \mathbf{K} = \int_\Omega [\mu W_{,m} U_{i,m} + (\lambda + \mu) W_{,i} U_{m,m}] \, \mathrm{d}\Omega.$$

M is the mass matrix, **K** is the stiffness matrix and W is a weighting function. For the Galerkin finite element method the weighting function is the same as the shape function. In this paper, in order to directly obtain three dimensional mode shapes for all the modes in the investigated frequency range, the cylinder is modeled as a collection of three dimensional block elements with eight nodes and three translation degrees of freedom per node. Thirty six elements and six layers were used in the circumferential and the longitudinal directions, respectively.

The cylindrical model used in the calculations is shown in Figure 1: $L = 522{\cdot}9$ mm, $R1 = 238{\cdot}05$ mm, $R2 = 158{\cdot}45$ mm. The model is made of mild steel. The material constants used in the calculation are: modulus of elasticity $E = 207$ GPa, density $\rho = 7800$ kg/m^3, and Poisson's ratio $\nu = 0{\cdot}30$. The calculated natural frequencies of the first 21 modes are presented in Table 1. The software ANSYS (education version) was used to perform the finite element analysis. The subspace method and frequency shift technique were used in the mode extraction. The calculated natural frequencies and mode shapes, in the frequency range 0–10 kHz, are listed in Table 2.

3. CLASSIFICATION OF MODES

Mode shapes of a thick cylinder are three-dimensional, and all three components of displacement may be associated with each resonant frequency. A pure radial excitation will give rise to a response not only in the radial direction, but to one which has components in the two other orthogonal directions. A classification of the modes of a thin cylinder, given by reference [13], with the further addition of a circumferential mode category, can be used for the thick cylinder, i.e. the modes of a thick cylinder can be classified using the following categories.

A. Pure radial modes: the vibration of this kind of mode is primarily due to pure radial motion and the cylinder retains a constant cross-sectional shape along its length, further the cross section remain plane and normal to the cylinder axis. An example is shown in row A of Table 2. These modes are important because the lowest mode of a thick cylinder is a pure radial mode.

B. Radial motion with radial shearing modes: for this kind of mode, the cylinder no longer retains a constant cross-sectional along its length as in the previous case. The circumferential cross sections do not remain plane, the generatrices do not remain parallel to each other, and for higher modes they are no longer straight lines. Examples are shown in row B in Table 2.

TABLE 1

Natural frequencies for the test cylinder

Mode	Resonant frequency (Hz) Experimental	Calculated	Error %	n	m	Mode type
1	1240	1251	0·9	2	0	A: pure radial mode
2	1432	1451	1·3	2	1	B: radial motion with shearing mode
3	3032	3038	0·2	1	1	E: axial bending mode
4	*	3090	—	0	1	F: global torsion mode
5	3136	3240	3·2	2	2	B: radial motion with shearing mode
6	3200	3266	2·0	3	0	A: pure radial mode
7	3432	3520	2·5	3	1	B: radial motion with shearing mode
8	3456	3515	1·7	1	2	F: global bending mode
9	3944	3918	−0·7	0	0	C: extensional mode
10	4152	4129	−0·6	0	1	C: extensional mode
11	4296	4282	−0·3	0	2	C: extensional mode
12	4472	4632	3·5	3	2	B: radial motion with shearing mode
13	4632	4741	2·3	2	1	E: axial bending mode
14	4784	4788	0·1	1	2	E: axial bending mode
15	5256	5621	6·5	1	3	F: global bending mode
16	5296	5616	5·7	0	3	C: extensional mode
17	5512	5720	3·6	4	0	A: pure radial mode
18	5552	5603	0·9	0	1	F: global longitudinal mode
19	5728	5960	3·9	4	1	B: radial motion with shearing mode
20	6080	6140	1·0	1	1	D: circumferential mode
21	6125	6211	1·4	2	2	E: axial bending mode

Error % = $(f_{cal.} - f_{exp.})/f_{exp}$ %. *: Not measured.

C. Extensional modes: for these modes, the median surface of the thick cylinder is stretched in the circumferential direction. Some authors refer to these as the breathing type modes. An example is given in row C in Table 2.

D. Circumferential modes (new category): in these modes, adjacent segmental elements expand or contract one by one in the circumferential direction. The median circumferential length of an expanding segment becomes longer and the length of the contracting segment becomes shorter. A nodal radius exists when there is no net circumferential displacement of a segmental element. An example is given in row D in Table 2.

E. Axial bending modes: for these modes, the circumferential cross-section is divided into several segments; adjacent segments bend oppositely in the axial direction, hence, nodal radial lines exist in the circumferential cross-section where the curvature of the transverse plane is zero. See row E in Table 2 for examples.

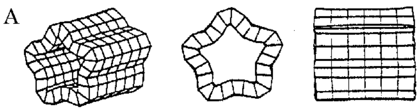
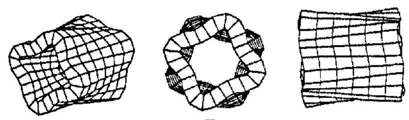
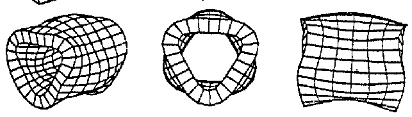
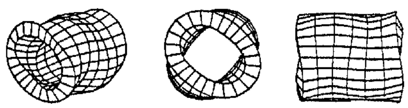
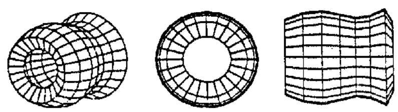
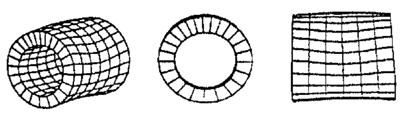
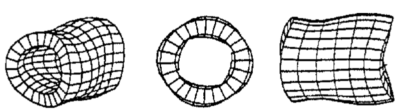
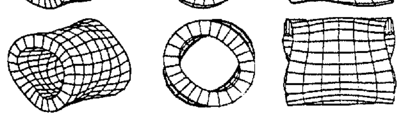
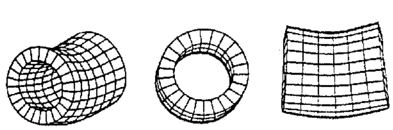
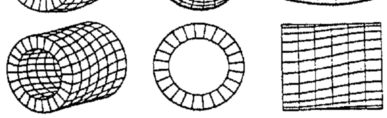
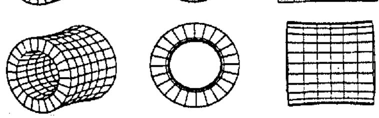
F. Global modes: for these modes the thick cylinder can be considered to behave as one of the following: a simple beam vibrating in a transverse direction, a bar vibrating in torsion, or as a rod vibrating in a longitudinal direction, and these are respectively shown as examples a, b and c of row F in Table 2.

In Table 2, n is the mode order in the circumferential direction, and the number of nodal radii of the nth mode is $2n$; m is the mode order in the axial direction, and the number of nodal cross sections or nodal median circles of the mth mode is m. Table 3 shows 5 different cross sections, which are equally spaced along the length of the cylinder, for the three modes $n = 3$, and $m = 0$, 1, and 2, respectively.

Tables for the natural frequencies of the same thick cylinder model, computed using the energy method, based on the three-dimensional theory of elasticity, are given in reference

TABLE 2

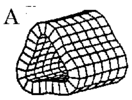
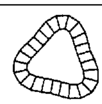
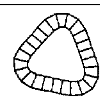
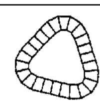
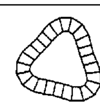
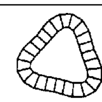
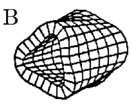
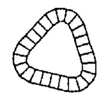
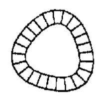
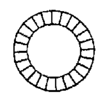
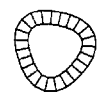
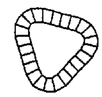
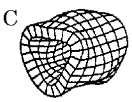
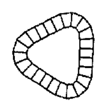
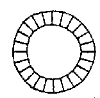
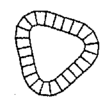
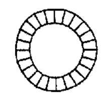
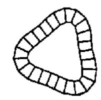
Classification of vibrational modes of thick cylinders

Mode shape	n	m	Frequency
A 	2	0	1251
	3	0	3266
	4	0	5720
	5*	0	8445
B 	2	1	1451
	3	1	3520
	4*	1	5960
	5	1	8653
	2	2	3240
	3*	2	4632
	4	2	6844
	5	2	9371
	2*	3	6216
	3	3	7875
	4	3	8524
	5	3	10771
C 	0	0	3918
	0	1	4129
	0	2	4282
	0*	3	5616
	0	4	8847
D 	1	1	6140
	1*	2	8219
	1	3	9933
	2	1	8605
	2	2	9173
	3	1	8867
E 	1	1	3038
	2*	1	4741
	3	1	6597
	4	1	10259
	1	2	4788
	2*	2	6211
	3	2	8040
	4	2	10132
F (a) 	1*	2	3515
	1	3	5621
	1	4	9103
	1	5	out of range
(b) 	0*	1	3090
	0	2	6391
	0	3	10106
	0	4	out of range
(c) 	0	1	5603
	0*	2	10908
	0	3	out of range
	0	4	—

*: The mode shape illustrated.

TABLE 3

Mode shapes in the axial direction for the modes $n = 3$

Mode shape	m	Cross-section 1	2	3	4	5
A 	0					
B 	1					
C 	2					

[11]. In these tables the mode shapes are described by the number of circumferential nodes n, and the number of longitudinal nodes m, where n was determined theoretically. The value of m was theoretically assigned an even or odd unknown integer value, which was then determined experimentally using a single-axis accelerometer monitoring deflections in the radial direction only, along a given external generatrix. The values of n and m, thus assigned, corresponded to those which would be obtained by using projections, in both end view and plan, of the true deflected shape of the cylinder. From these tables it can further be seen that several natural frequencies can have the same numbers n and m, i.e. the same apparent mode shape. Table 4 shows some examples extracted from reference [11]. Actually the modes are quite different according to the mode classification of thick cylinders, as presented in this paper. In addition to the assigned values of n and m, and the new classification of modes, Table 4 also gives both the experimental frequency data and frequency data calculated by the finite element method.

4. EXPERIMENTAL RESULTS

To verify the calculated results, measurements were made on the experimental model which is exactly the same as the cylinder model shown in Figure 1. In order to extract the resonant frequencies and the mode shapes of the model, the frequency response functions were measured on all surfaces of the model including the end surfaces. In the test a hammer was used as the exciter, therefore, only those modes with natural frequencies up to 6·4 kHz could be obtained. Figure 2 shows a typical frequency response function of the experimental model. The first 21 resonant frequencies, and corresponding mode type descriptions of the model are given in Table 1. Figure 3 shows 6 typical experimental modes of the model. From Figure 3 and Table 1 it can be seen that the results for the experimental and finite element methods natural frequencies are sufficiently accurate to confirm the identity of the resonant frequencies, thus permitting cross-referencing of the results and those of reference [11]. Any errors in natural frequency data may be attributed to the finite element analysis. Because the educational version of ANSYS was used, a limit existed on the size of the wave front. By increasing the number of elements, better results are obtainable with the element used. Even better results can be obtained in the case of axisymmetric bodies, such as cylinders, by using a two dimensional

TABLE 4

Classification of the modes of a thick cylinder in the range 0–10 kHz

Mode n	m	Experimental* f(Hz)	Energy method* Calc. f(Hz)	Energy method* S-AS**	Present method Calc. f(Hz)	Mode description
1	2	3456	3414	S	3515	F: global bending mode
1	2	4778	4729	S	4788	E: axial bending mode
1	2	8198	8136	S	8219	D: circumferential mode
2	1	1432	1413	AS	1451	B: radial motion with shearing mode
2	1	4631	4573	AS	4741	E: axial bending mode
2	1	8441	8358	AS	8605	D: circumferential mode
2	2	3139	3107	S	3240	B: radial motion with shearing mode
2	2	6125	6064	S	6211	E: axial bending mode
2	2	9180	9049	S	9173	D: circumferential mode

*: From reference [11].
**: Energy method mode description; S-symmetric longitudinal mode. AS-antisymmetric-longitudinal mode.

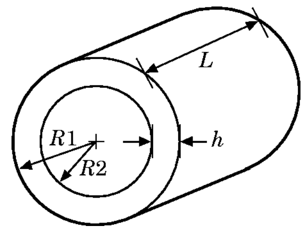


Figure 1. Calculated and experimental cylinder model.

element in the axial plane combined with a Fourier expansion in the circumferential direction. Since a uniaxial accelerometer was used for the input to one channel of the analyzer, the experimental system was incapable of measuring a torsional response, therefore, the pure torsional mode 4 was not detected, also the experimental mode shapes would not indicate shear deflections.

5. EFFECT OF AXIAL LENGTH

In order to further study the mode classification, the effects of varying axial length and varying radial thickness were investigated. A set of 8 thick cylinder models, comprising group 1, the general form of which is shown in Figure 1, were analyzed using the finite element method. The inside and outside radii of all the models were the same, $R1 = 114{\cdot}3$ mm and $R2 = 76{\cdot}2$ mm, and the length of each model is different and is as given in Table 5. All the models were modeled as collections of three-dimensional block elements with eight nodes, each of which had three translational-degrees-of-freedom. The models were made of mild steel, and the elastic constants were the same as used for the mode classification. Modes up to 20 kHz were studied. The calculated natural frequencies and mode shapes of the 8 models are given in Table 6 together with the classification category as proposed. Rows A to F represent pure radial modes, radial shearing modes, extensional modes, circumferential modes, axial bending modes and global modes, respectively.

To verify the calculated results, extensive experiments were performed on the experimental models which were exactly the same as the analytical models listed in Table 1.

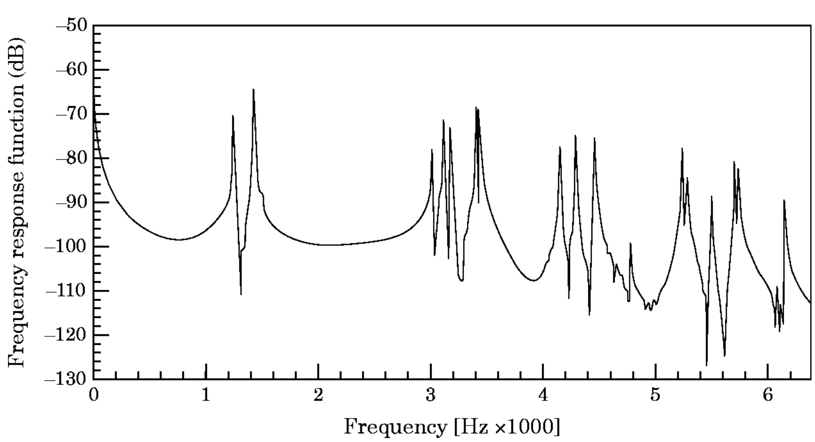


Figure 2. Frequency response function of cylinder model.

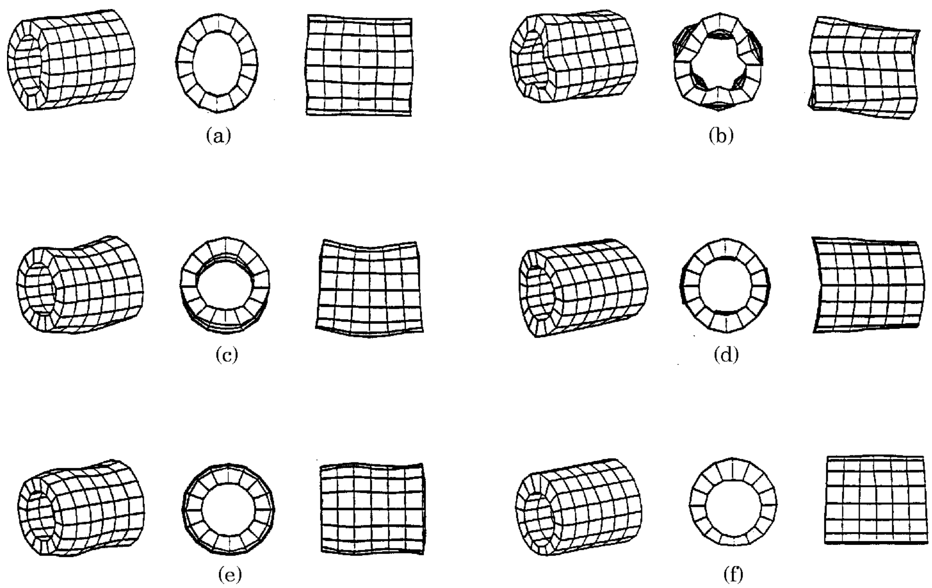


Figure 3. Typical experimental mode shapes: (a) mode 1, pure radial mode, $f = 1240$ Hz, $n = 2$, $m = 0$; (b) mode 7, radial motion with shearing mode, $f = 3432$ Hz, $n = 3$, $m = 1$; (c) mode 8, global bending mode, $f = 3456$ Hz, $n = 1$, $m = 2$ (d) mode 13, axial bending mode, $f = 4632$ Hz, $n = 2$, $m = 1$; (e) mode 16, extensional mode, $f = 5296$ Hz, $n = 0$, $m = 3$; (f) mode 20, circumferential mode, $f = 6080$ Hz, $n = 1$, $m = 1$.

To simulate free boundary conditions, the models were suspended by a rubber rope, in such a way that the mounting did not affect the vibration of the model. In order to measure the resonant frequencies of the models a random excitation signal was used. The frequency response functions were measured at several points on the surfaces of the model, including the end surfaces, in order to prevent possible mode missing. For the purpose of confirming the results of the analysis, the input frequency band of the random signal was the same as used in the calculations (0–20 kHz). From the frequency response functions the resonant frequencies of the model could be obtained. To identify the mode shape of the vibration associated with a particular resonant frequency, the relative amplitudes and phases of vibration were measured at a sufficient number of points on all surfaces. The experimental resonant frequencies for the 8 models are also given in Table 6. The different types of modes follow their own rules, with respect to axial length and radial thickness, as discussed below.

TABLE 5

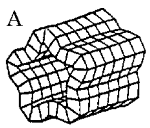
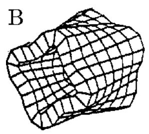
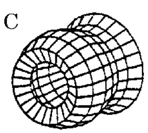
Dimensions of the series of cylinders in group 1

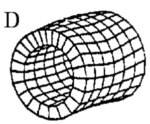
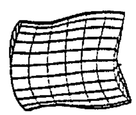
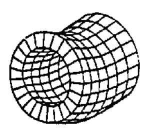
Model	$R1$ (mm)	$R2$ (mm)	R (mm)	L (mm)	h (mm)	L/h	h/R
1	114·3	76·2	95·3	12·7	38·1	0·33	0·4
2	114·3	76·2	95·3	25·4	38·1	0·67	0·4
3	114·3	76·2	95·3	50·8	38·1	1·33	0·4
4	114·3	76·2	95·3	76·2	38·1	2·00	0·4
5	114·3	76·2	95·3	101·6	38·1	2·67	0·4
6	114·3	76·2	95·3	127·0	38·1	3·33	0·4
7	114·3	76·2	95·3	177·8	38·1	4·67	0·4
8	114·3	76·2	95·3	254·0	38·1	6·67	0·4

$R = (R1 + R2)/2 \qquad h = R1 - R2.$

TABLE 6

Analytical and experimental natural frequencies for group 1

Mode	*n*	*m*	Mdl. 1	Mdl. 2	Mdl. 3	Mdl. 4	Mdl. 5	Mdl. 6	Mdl. 7	Mdl. 8
	2	0	2521	2522	2527	2531	2551	2563	2580	2594
			2521†	*2522*	*2527*	*2536*	*2542*	*2551*	*2556*	*2570*
A 	3	0	6626	6630	6646	6683	6718	6743	6776	6800
			6438	*6456*	*6464*	*6486*	*6504*	*6533*	*6552*	*6588*
	4	0	11696	11704	11740	11807	11859	11894	11935	11966
			11097	*11132*	*11187*	*11247*	*11300*	*11335*	*11381*	*11417*
	5*	0	17414	17428	17482	17577	17637	17677	17721	17756
			16179	*16243*	*16302*	*16387*	*16452*	*16507*	*16553*	*16597*
	2	1	889	1683	2830	3326	3449	3409	3218	2989
B 			*842*	*1664*	*2723*	*3233*	*3375*	*3351*	*3179*	*2962*
	3	1	2465	4552	7269	8207	8283	8083	7657	7291
			2455	*4400*	*6496*	*7922*	*8083*	*7862*	*7454*	*7100*
	4*	1	4691	8329	12479	13623	13593	13291	12790	12433
			4668	*7850*	*11777*	*12980*	*12971*	*12698*	*12224*	*11861*
	5	1	7654	12831	18106	19321	19241	18927	18458	18131
			7602	*11872*	*16817*	*18111*	*18050*	*17739*	*17287*	*16966*
	0	0	8617	8714	8706	8693	8671	8640	8530	8081
C 			*8742*	*8752*	*8750*	*8737*	*8719*	*8692*	*8583*	*8149*
	0	1	2791	4868	6977	7792	8154	8338	8494	8567
			2783	*4875*	*6975*	*7823*	*8182*	*8347*	*8542*	*8615*
	0	2					17369	13175	9916	8845
							16505	*12831*	9885	8866
	0*	3			out of range				16340	11065
									15289	*10874*
	0	4								16013
										15030

	1	0	11887	11883	11863	11827	11768		do not exist	
			11817	*11892*	*11884*	*11854*	*11798*			
D 	1	1					18889	16272	13814	12647
							18752	*16271*	*13837*	*12616*
	1*	2							15264	12853
					out of range				*15102*	*12809*
	1	3								19360
										18979
	1*	1	4472	7166	8663	8523	8174	7783	7044	6240
			3712	*6878*	*8428*	*8431*	*8130*	*7768*	*7067*	*6286*
E 	2	1	7672	11729	12822	11656	10912	10518	10161	9657
			7104	*10985*	*12120*	*11266*	*10719*	*10397*	*10108*	*9588*
	3	1	11390	16927	17902	16086	15264	14974	do not exist	
			10240	*15620*	*16691*	*15395*	*14856*	*14683*		
	4	1	15551				do not exist			
			14352							
F 	1*	2					17964	13671	9769	7109
							17008	*13242*	*9657*	*7104*
	1	3							16881	11064
					out of range				*15774*	*10803*
	1	4								16127
										15201

*: The mode shape illustrated.
† Italic figures indicate experimental results.

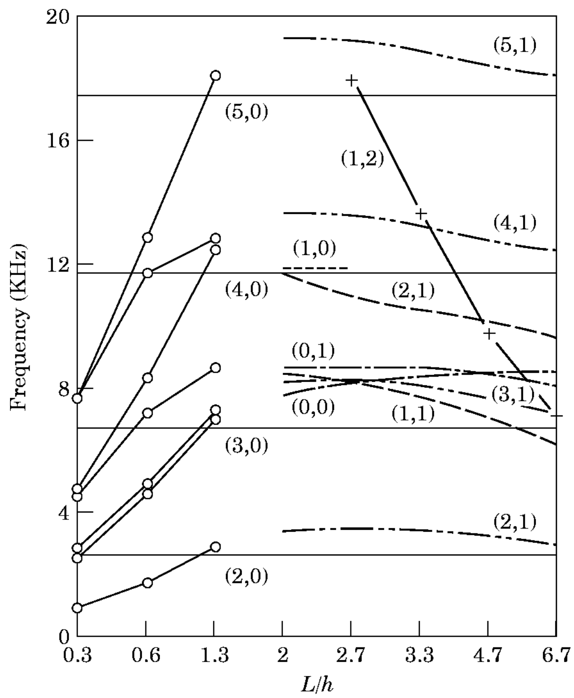


Figure 4. Effect of axial length on natural frequencies; designation (n, m), group 1, $h/R = 0{\cdot}4$. ——, A: pure radial modes; — - - —, B: radial motion with shearing modes; — - —, C: extensional modes; - - - -, D: circumferential modes; – – –, E: axial bending modes; — + —, F: global bending modes; ——, in-plane modes (plate); —○—, transverse modes (plate). At low L/h ratios, the cylinder behaves effectively as a plate.

To study the effect of the axial length on the modes of thick cylinders, the results are plotted as shown in Figure 4. From these plots, the following observations can be made:

1. The vibrational behavior of models 1 and 2 (short cylinders) are different from the other longer models except in the case of the pure radial modes. The lowest mode of models 1 and 2 is the radial and shearing mode ($n = 2$, $m = 1$). The reason for this is that models 1 and 2 are not true cylinders; because their axial length is shorter than their radial thickness, they are actually annular plates. Hence, from the point of view of vibration, the difference between an annular plate and a thick cylinder is to be found in the form of the lowest mode, for the former it is a radial motion with radial shearing mode (row B of Table 6), whilst for the latter it is a pure radial mode (row A of Table 6). In other words, for the range of models discussed here, the ratio of the axial length L to the radial thickness h can be divided into three ranges, the plate range $L/h \leqslant 1$, the thick cylinder range $L/h \geqslant 2$ and the transition range $1 \leqslant L/h \leqslant 2$.

2. For a thick cylinder all the modes can be divided into two categories, axial length independent modes and axial length dependent modes. All the pure radial modes are axial length independent modes, and their natural frequencies are dependent only upon their radial dimensions. Hence, even the natural frequencies of such modes for models 1 and 2, short cylinders, are the same as those for the long thick cylinders. Similar frequencies are also found in annular plates of the same radial dimensions. The common characteristic of such modes is $m = 0$, i.e. similar to the in-plane vibrations of a plate.

All other kinds of modes of thick cylinders are axial length dependent modes, and their natural frequencies are not only dependent upon the radial dimensions, but are also dependent upon the axial length, and hence, the L/h ratio when h is constant. Generally, the natural frequency decreases in a non-linear manner as the length of the cylinder increases. In all kinds of axial length dependent modes, the natural frequencies of the global modes are the most sensitive to length changes (row F of Table 6). For other kinds of axial length dependent modes, as the number of circumferential nodes n increases, the natural frequencies become more sensitive to changes in axial length. The extensional mode $m = 1$ (row C of Table 6) is an exception; the natural frequency continues to increase as the length of the thick cylinder increases.

3. As the axial length of a thick cylinder increases, the mode shapes of the pure radial modes are retained. For the axial length dependent modes, the mode shape may change, and the longitudinal node number m may increase. For example, when the length of the cylinder becomes long enough, there is no circumferential mode with $n = 1, m = 0$.

4. Each category of modes for a thick cylinder shows different rates at which their natural frequencies decrease with increasing L/h ratio. For the pure radial modes and radial motion with shearing modes, the rate is low, and these modes can be found in all the models (even those which are effectively plates). For this reason these modes are the most important modes of a thick cylinder, especially the pure radial mode $n = 2$, which is the lowest mode for all thick cylinders.

5. The models with $L/h \leqslant 1$ belong to the category of annular plates because their lowest modes are the transverse vibration modes as discussed above. For an annular plate, it is known that all the modes of the plate can be classified as either transverse vibrational modes, or in-plane vibrational modes. From Figure 4 it can be seen that the transverse vibrational modes are axially length dependent, and generally the natural frequency of a mode increases as the axial length increases, which is opposite to the behaviour of the axial length dependent modes of thick cylinders. The in-plane vibrational modes are axial length independent modes, and their natural frequencies are dependent upon only the radial size of the model; it can further be seen from Figure 4 that their natural frequencies are the same as those of thick cylinders. Hence, an annular plate can be used as a model to predicate the natural frequencies of the pure radial modes of a thick cylinder.

6. EFFECT OF RADIAL THICKNESS

In order to study the effect of radial thickness on the different modes, the natural frequencies and mode shapes were calculated for a further three groups of cylinders (groups 2, 3, and 4) which had the same outside radius as that of group 1, but different inner radii; see Table 7 for details. Each group consisted of 8 models whose length were the same as those of the corresponding 8 models of group 1. The calculated results for group 3, $h/R = 0{\cdot}2$, are plotted in Figure 5. Similar calculated results were obtained for groups 2 and 4. From the calculated results the following conclusions can be drawn:

1. All of the previous conclusions with respect to the effects of axial length on the different modes, obtained for group 1, are valid for all groups. As the L/R ratio decreases from 0·4 to 0·1 there is a corresponding change in the location of the transition range from plate to thick cylinder, the value for L/h increasing from 1·0 to 2·0.

2. As the radial thickness decreases, i.e. the thick cylinders become thinner, the natural frequencies of all corresponding modes decrease, the rate of decrease is different for each mode. Figure 6 shows the relation between natural frequencies of different kinds of modes with varying radial thickness for the longest model 8 from all four groups. It can be seen that for all modes there exists an approximately linear relationship, but the slopes are different. All the pure radial modes, which are independent of axial length change, and

TABLE 7

Sectional properties of groups of models with equal outside radius R1

Group	$R1$ (mm)	$R2$ (mm)	R (mm)	h (mm)	h/R
1	114·3	76·2	95·3	38·1	0·4
2	114·3	84·5	99·4	29·8	0·3
3	114·3	93·5	103·9	20·8	0·2
4	114·3	103·4	108·9	10·9	0·1

$R = (R1 + R2)/2 \qquad h = R1 - R2.$

the radial motion with shearing modes are very sensitive to radial thickness changes. The slopes of equal order modes for these two categories are almost the same. All other types of modes, which are sensitive to axial length change, as discussed earlier, are relatively insensitive to a change in radial thickness.

3. The vibrational behaviour of the pure radial modes is different from that of all the other kinds of modes. The natural frequencies of such modes are only dependent upon the radial dimensions of the model, and the natural frequencies of all other kinds of modes are dependent upon both the axial and radial dimensions. All the modes of a thick cylinder, therefore, can be separated into either pure radial modes, or non pure radial modes.

To further study the effect of radial thickness on the different modes of thick cylinders, the natural frequencies and mode shapes of another four thick cylinders were also analyzed and the results are shown in Figure 7. The difference between these four models and those in groups 1 to 4 is the outside radius of the latter groups was constant, while for the former

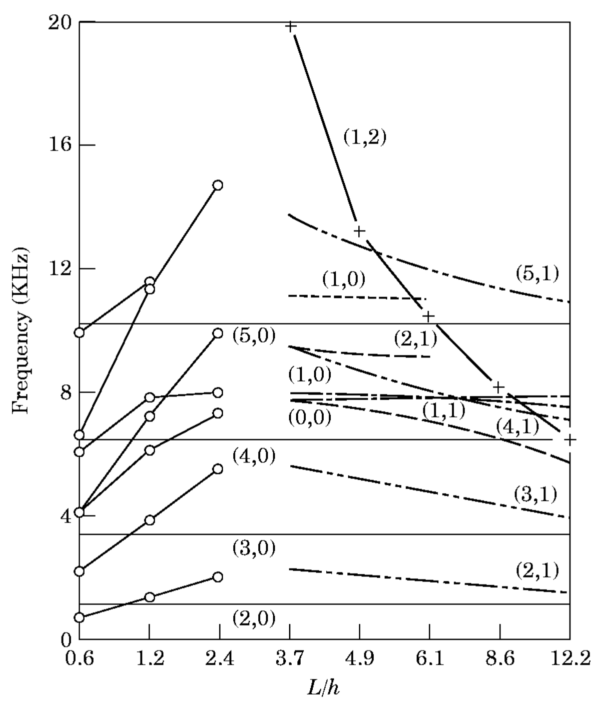


Figure 5. Effect of axial length on natural frequencies, designation (n, m), group 3, $h/R = 0{\cdot}2$. ——, A: pure radial modes; — - - —, B: radial motion with shearing modes; — - —, C: extensional modes; - - - -, D: circumferential modes; – – –, E: axial bending modes; — + —, F: global bending modes; ——, in-plane modes (plate); —○—, transverse modes (plate). L/h ratio comment as in Figure 4.

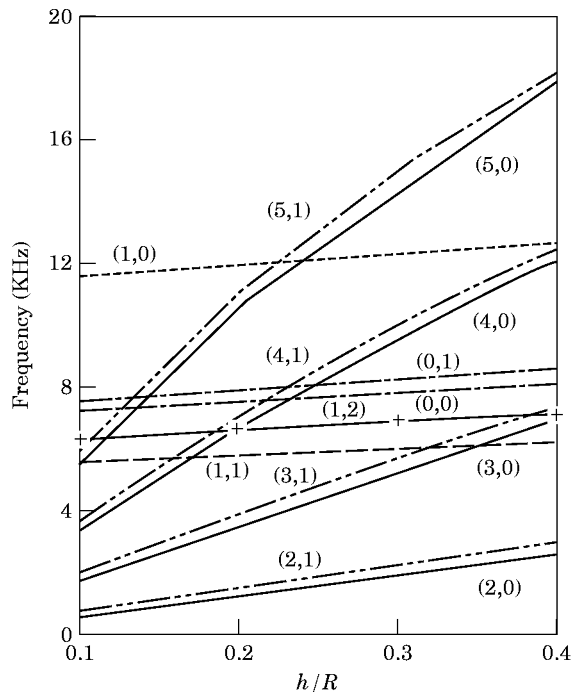


Figure 6. Effect of radial thickness on natural frequencies, designation (n, m), $L = 522$ mm, $R1$ = const. ——, A: pure radial modes; — - - —, B: radial motion with shearing modes; — - —, C: extensional modes, - - - -, D: circumferential modes; – – –, E: axial bending modes; — + —, F: global bending modes.

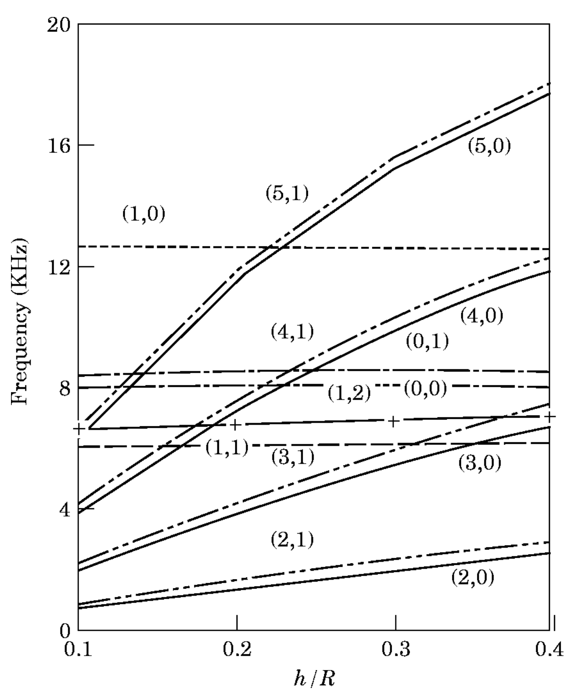


Figure 7. Effect of radial thickness on natural frequencies, designation (n, m), $L = 522$ mm, median R = Const. ——, A: pure radial modes; — - - —, B: radial motion with shearing modes; — - —, C: extensional modes; - - - -, D: circumferential modes; – – –, E: axial bending modes; — + —, F: global bending modes.

TABLE 8

Sectional properties of models with constant median radius R

Model	$R1$ (mm)	$R2$ (mm)	R (mm)	h (mm)	h/R
1	114·3	76·2	95·3	38·1	0·4
2	109·6	80·9	95·3	28·5	0·3
3	104·8	85·7	95·3	19·1	0·2
4	100·0	90·5	95·3	9·5	0·1

$R = (R1 + R2)/2$ $\quad h = R1 - R2$.

one the median radius was constant, as shown in Table 8. It can be seen that all the conclusions regarding the effects of radial thickness remain valid for this cylinder group, however, all their natural frequencies are higher than those of the group shown in Figure 6. Hence, the natural frequencies of thick cylinders are dependent upon not only the ratio of the radial thickness to the median radius of the thick cylinder, but also upon the absolute size of the cross section of the cylinder.

7. CONCLUSIONS

Based on the three-dimensional mode shapes, a mode classification for thick cylinders is presented in this paper. The use of a simpler descriptor, based on the numbers of circumferential nodes n and longitudinal modes m, is not sufficient for identifying a specific mode. Used in conjunction with any one of the 6 classes of vibration, n and m can be interpreted exactly to specify the mode shape of a given mode of vibration.

All types of modes can be further subdivided into either pure radial modes, or non-pure radial modes. The natural frequencies of the pure radial modes can be considered to be independent of axial length, but dependent upon the radial dimensions, and generally the thicker the cylinder wall, the higher the natural frequencies of such modes. The natural frequencies of non-pure radial modes are dependent upon both the axial length and radial dimensions. Generally, the longer the model, the lower the natural frequencies; and the thicker the cylinder wall, the higher the natural frequencies.

The pure radial modes and the radial motion with shearing modes are the most important from an engineering standpoint. This is because of their low frequencies, and because they always exist in any cylinder, whether it is long or short. The lowest mode of a thick cylinder is the pure radial mode $n = 2$, this is usually the one of most significance from an engineering standpoint. Because the classification is based on the nature of the mode shapes of the cylinder, it may provide a basis for the development of displacement function which better describe these true mode shapes.

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