

Table 1.4. *The simple roots and fundamental weights for the classical algebras*

Algebra	Simple root α_i	Fundamental weight ω_i
A_r	$e_i - e_{i+1}, 1 \leq i \leq r$	$\sum_{j=1}^i e_j - \frac{i}{r+1} \sum_{j=1}^{r+1} e_j$
B_r	$e_i - e_{i+1}, 1 \leq i < r$ $2e_r$	$e_1 + \cdots + e_i, 1 \leq i < r$ $\frac{1}{2}(e_1 + \cdots + e_r)$
C_r	$\sqrt{2}(e_i - e_{i+1}), 1 \leq i < r$ $\sqrt{2}e_r$	$\frac{1}{\sqrt{2}}(e_1 + \cdots + e_i), 1 \leq i \leq r$
D_r	$e_i - e_{i+1}, 1 \leq i < r$ $e_{r-1} + e_r$	$e_1 + \cdots + e_i, 1 \leq i < r - 1$ $\frac{1}{2}(e_1 + e_2 + \cdots + e_{r-2} + e_{r-1} - e_r), i = r - 1$ $\frac{1}{2}(e_1 + e_2 + \cdots + e_r), i = r$

(inner automorphisms) of \mathfrak{g} , and the resulting diagram is uniquely determined. This is a powerful paradigm: to understand and classify a rigid structure, find and study a combinatorial characterisation. Later we apply this strategy to conformal field theories.

These choices though should disturb the mathematician in us. Perhaps the presence of the Weyl group in the following is a hint that we are doing Lie theory badly. Just as the vector space ‘symmetry’ GL_n is the artificial consequence of choosing a basis, so is the Weyl group the bad karma caused by selecting one positive chamber over all others. Probably an approach based on Vogel’s universal Lie algebra (Section 1.6.2) will ultimately be preferable.

In any case, we are most interested in the Killing form and Weyl group restricted to \mathfrak{h}^* . Given simple roots α_i , define *fundamental weights* $\omega_i \in \mathfrak{h}^*$ to be the dual basis ($\omega_i | \alpha_j = \delta_{ij}$). They lie on the edges of the chamber C . Their \mathbb{Z} -span is the lattice dual to the root lattice, called the *weight lattice*. Denote by P_+ the intersection of the weight lattice with C , so $\lambda \in P_+$ if and only if $\lambda = \sum_{i=1}^r \lambda_i \omega_i$ where each *Dynkin label* λ_i lies in \mathbb{N} . These $\lambda \in \mathbb{N}$, called *dominant integral weights*, are the r -tuples of Theorem 1.5.1.

Table 1.4 gives the α_i and ω_i for the classical algebras, using an orthonormal basis of \mathbb{R}^r (\mathbb{R}^{r+1} for A_r). Nodes are labelled as in Figure 1.17 – this is the labelling used in, for example, [328] but not by all other authors. The table makes manifest the Killing form on \mathfrak{h}^* , and is useful in the study of affine Kac–Moody algebras (Section 3.2). More data for the simple Lie algebras, including the exceptional ones (avoided here for reasons of brevity), can be found in section 6.7 of [328], chapter 7 of [214], and especially pages 265–90 of [84].

The Weyl group of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ is the symmetric group S_n and acts on \mathfrak{h}^* by permuting the subscripts: $\sigma \sum_i h_i \omega_i = \sum_i h_i \omega_{\sigma i}$. Figure 1.18 gives the root systems of the semi-simple Lie algebras of rank 2. A choice of simple roots is indicated by the numerals ‘1’ and ‘2’. In Figure 1.19 a portion of the weight lattices of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ are displayed, along with simple roots and fundamental weights, and the Weyl reflections $r_i = r_{\alpha_i}$ through the simple roots. Note the $S_2 \cong \{\pm 1\}$ symmetry of the A_1 weight lattice, and the S_3 symmetry of the A_2 weight lattice.