

ZERO TO THE ZEROTH POWER

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1. PROLOGUE

You have probably never seen the Loch Ness monster. On the other hand, it would be rather surprising if you had never come across the Power Less monster in the course of an exercise, or when a curious student asked you a question, or when another less well-intentioned student tried to embarrass you in front of the class.

We no longer bother to count the number of appearances of Power Less on the Web in mathematical forums, where he surfaces again and again, bringing with him the same questions and perpetual controversies.

But what is this mysterious sea serpent? No, it is not $\frac{0}{0}$, now only too well known. Rather, it is one of its close relatives, the strange 0^0 , zero to the zeroth power.

Certain people will tell you that 0^0 is indeterminate. Others think that $0^0 = 0$ in certain cases. Still others will contend that $0^0 = 1$, or more prudently that this equality is a “convention.” However, if this “equality” is nothing more than a convention and not a proven statement, how are we to know in which contexts it holds true?

2. A SET THEORETIC POINT OF VIEW

Let us consider two sets N and M consisting of n and m elements, respectively (i.e., $\text{card}(N) = n$ and $\text{card}(M) = m$). The cardinality of the set of functions from N to M (that is, the number of functions) is m^n .

In the particular case where $n = m = 0$, we have the number of functions from the empty set to itself, which yields one single function. Conclusion: $0^0 = 1$.

This is not terribly intuitive! Nevertheless, this conclusion is logical and coherent in the context of combinatorics, among others. Thus, in this case, $0^0 = 1$ is more than a mere convention: it is a proven equality.

3. AN ELEMENTARY ALGEBRAIC POINT OF VIEW

Given a nonzero real number x and a positive integer n , what is the meaning of x^n ? Every one of us learned at some point or another that x^n is obtained by multiplying x by itself n times, that is $x^n = x * x * \cdots * x$, where x appears n times on the right-hand side.

Let m be an integer, possibly 0. According to what we just saw, $x^{n+m} = x * x * \cdots * x$, where x appears $n+m$ times, which is permissible since $n+m > 0$. We may split this product of x 's into two parts, one (x^n) containing n copies of x and the other (x^m) containing m copies. Thus, in the case where $m = 0$, this second term contains zero copies of x . Its value, formally written x^0 , is therefore necessarily 1 because in this case, $x^n * 1 = x^n = x^{n+0} = x^n * x^0$.

We might be tempted to use the same reasoning for the case $x = 0$, which would yield $0^0 = 1$. However, returning to the beginning of this paragraph where we obtained x^n by multiplying x by itself n times, for $x = 0$ we have $0^1 = 0$, $0^2 = 0 * 0 = 0$, etc. Thus we have $0^n = 0$, which would lead us to expect $0^0 = 0$, contrary to the previous section!

The source of this dilemma clearly lies in our elementary definition of x^n being “obtained by multiplying x by itself n times,” which in the case $n = 0$ leads us to a statement devoid of meaning: “ x^0 is obtained by multiplying x by itself 0 times.” What could it possibly mean to multiply something zero times?

Certainly we have seen that we may extend this definition to $x^0 = 1$ for nonzero x , which is therefore not merely a convention, but a proven equality in this case. On the other hand, for $x = 0$, the convention could be $0^0 = 0$ just as well as $0^0 = 1$.

4. A TOPOLOGICAL POINT OF VIEW

The idea is to define, if possible, 0^0 as the limit of a function of two real variables x and y as they both approach 0. Therefore, we consider the function

$$f(x, y) = x^y = \exp(y \ln(x)), \quad \text{for } x > 0.$$

- In the case $y = 0$, we have $x^0 = \exp(0) = 1$.
- In the case $y > 0$, as x approaches 0, $y \ln(x)$ approaches $-\infty$ and x^y therefore approaches 0.
- In the case $y < 0$, as x approaches 0, $y \ln(x)$ approaches $+\infty$ and thus x^y approaches $+\infty$.

We see already that, depending on whether y is positive, negative, or zero, the hypothetical limit 0^y is not always the same. It jumps from 0 to $+\infty$ as y changes from negative to positive. However, this limit does yield $x^0 = 1$ for $y = 0$, making us suppose that $0^0 = 1$.

This conjecture is reinforced by the fact that, for $y = x$, the limit of $x^x = \exp(x \ln(x))$ is 1 as x approaches 0, since $x \ln(x)$ tends to 0.

Examining the preceding examples, we may be tempted to believe that an approach using limits will always yield $0^0 = 1$. However, things are not so simple. This limit also depends on the manner in which x and y approach 0, either independently from each other, or simultaneously. For instance, let us consider the function

$$f(x) = x^{\frac{2}{\ln(x)}} = \exp\left(\frac{2}{\ln(x)} \ln(x)\right) = \exp(2)$$

As x tends to 0, the exponent $y = 2/\ln(x)$ approaches 0. Thus, in the limit, $0^0 = e^2$.

And this other example:

$$f(y) = \left(\exp\left(-\frac{1}{y}\right)\right)^y = \exp\left(y\left(-\frac{1}{y}\right)\right) = e^{-1}$$

As y approaches 0, $\exp(-1/y)$ tends to 0. Therefore, in the limit, $0^0 = e^{-1}$.

These examples may seem far-fetched. Nonetheless, while calculating the limit of complicated functions, it is not uncommon to come across expressions that reduce down to x^y with x and y both approaching 0, yet whose limit (formally 0^0) is neither 1 nor 0.

Using this approach, we are therefore led to believe that 0^0 is an indeterminate form.

5. AND WHY NOT A GRAPHICAL APPROACH?

This is certainly not the most elegant approach, and theoreticians will probably not appreciate it. Nevertheless, for those for whom a

picture is worth a thousand words, figure 1 illustrates that which was discussed in the preceding paragraph.

In fact, this figure is a kind of abacus which allows us to read off the value approached by $f(x, y) = x^y$ corresponding to given x and y . If the point (x, y) does not fall exactly on one of these curves, the value approached by x^y may be obtained by interpolation. The precision of this graph is quite obviously not nearly as good as that of a graphing calculator, but this is unimportant for our purposes.

We are attempting to observe what occurs in the neighborhood of $(0, 0)$ in order to understand the meaning of 0^0 , in the limit. In this region the curves get closer and closer together, so much so that they become indistinguishable, leaving us in a state of suspense. Accordingly, we magnify this region.

The enlarged figure 2 is still not fully satisfactory, and we realize that, even if we were to pursue this approach further, making larger and larger magnifications, we still would be unable to distinguish the different curves as they approach the origin.

It is clear that in the immediate neighborhood of the origin, all the curves become indistinguishable, which, visually and thus intuitively, tells us that, for x positive approaching 0 and y approaching 0, the limit of x^y could take on any positive value, even 0 or $+\infty$.

If this isn't indeterminate, then what is?

6. EPILOGUE

Hopefully everyone will have understood that, if the branch of mathematics in which we are working is not specified, then 0^0 is indeterminate. On the other hand, if we are working in a specific context, it is often possible to give a rigorous meaning to the symbols 0^0 .

Very frequently the convention $0^0 = 1$ allows us to simplify formulae and to avoid treating several cases separately.

The formalism $0^0 = 1$ is more than a convention and may be used as a proven identity in an explicitly indicated context. However, this may lead to ambiguity, and we should, with prudence as our guide, consider this convention as a convenient shorthand, rather than a general identity. If there is a domain where mistrust, or even distrust, is the rule,

it is that of the calculation of limits and the study of local behavior in the neighborhood of singularities.

Have we therefore flushed out said Power Less monster? Yes, certainly, and we are by no means the first to do so. All this has been known for years, which does not prevent its repeated reappearances. Flushed out, he is and has been. But extinct, annihilated, nay! For this to be the case, 0^0 must be as common and as well known as its ancestor $\frac{0}{0}$.

But why do we continue to call this other beast Power Less's "ancestor?" Well, please excuse this last exercise in sleight of hand:

Let $t = 1/\ln(x)$. For x and y approaching 0, $\ln(x)$ approaches $-\infty$ and t therefore tends to 0. We have $x^y = \exp(y \ln(x)) = \exp(y/t)$, with y and t both approaching 0. And thus we discover the aforementioned $\frac{y}{t} = \frac{0}{0}$.

What! 0^0 and $e^{\frac{0}{0}}$ are therefore equivalent?

Seriously, let us not delude ourselves: we have not seen the last of the Power Less monster. As for myself, I have not heard the last of it from my Scottish friends, who continue to reproach me for this implausible comparison between their renowned monster and a vulgar $\frac{0}{0}$ gussied up with an extravagant name.