CHAPTER 3

Plunging

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Ever since Francis Bacon, it had been believed that the laws of Nature were there to be "discovered," if only one made the right experiments. Einstein taught us differently. He stressed the vital role of human inventiveness in the process. Newton "invented" the force of gravity to explain the motion of the planets. Einstein "invented" curved spacetime and the geodesic law [that describes the path of a free particle]; in his theory there is no *force of gravity. If two such utterly different mathematical models can (almost) both describe the same observations, surely it must be admitted that physical theories do not tell us what nature* is, *only what it* is like. *The marvel is that nature seems to go along with some of the "simplest" models that can be constructed in the context of various mathematical formalisms.*

— Wolfgang Rindler

1 GOING STRAIGHT

"Go straight!" spacetime shouts at the stone.
The stone's wristwatch verifies that its path is straight.

All the exotic talk about curved spacetime near stars and black holes leaves us unprepared for a revelation about motion right at home: Schwarzschild geometry correctly describes the motions of footballs and stones near Earth's surface. Even more surprising: Analyzing trajectories of near-Earth objects using Schwarzschild geometry prepares us to go back and describe trajectories around stars, white dwarfs, neutron stars, and black holes.

Why the particular parabolic path?

Throw a stone and let it fall back to Earth. In the uniform gravity near Earth's surface the stone follows a parabolic path in space, tracing out the solid curve in the diagram in the left panel of Figure 1. At the beginning and

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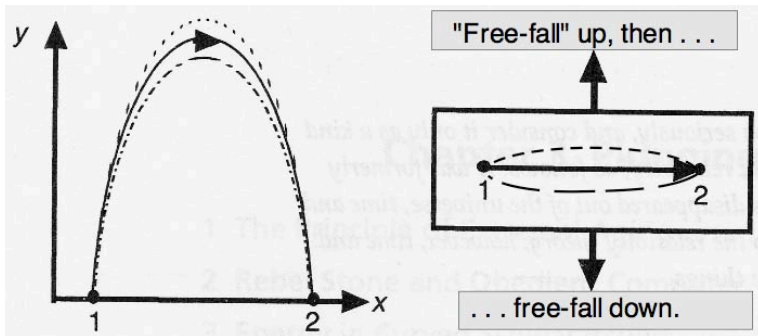

FIGURE 1 Parabolic path of a stone (solid line, left panel) connecting launch (Event 1) and impact (Event 2). Dashed lines show alternative spatial paths between these two events, alternatives that the stone does not take. On the right is a free-fall frame that rises and falls with the stone. With respect to this inertial frame, the stone follows a straight path (solid line, right panel). Plotting its motion as a function of time yields a straight worldline (Figure 2).

end of this path fix two events in space and time: Event 1, initial launch; Event 2, final impact. Why does the stone follow the particular path in space between Event 1 and Event 2, shown as a solid curve in Figure 1? Why not hurry faster along a higher, longer parabolic path, the upper dashed line in the figure, to get back in time for the appointed impact? Or move slower along a lower, shorter parabolic path, the lower dashed line? Why not take some entirely different trajectory between these two events? What command does spacetime give to the stone telling it how to move?

Spacetime to stone: “Go straight!”

Spacetime shouts, “Go straight!” The free stone obeys. What does “straight” mean? Straight with respect to what? We know the answer: The path of the stone is straight in all local free-fall (inertial) frames. Ride in an inertial frame that rises and falls vertically in concert with the stone, as shown in the right-hand panel of Figure 1. With respect to the free-fall frame the stone moves on a straight path during the entire trip between launch (Event 1) and impact (Event 2). None of the alternative trajectories on the left panel would be straight in the right panel.

“Straight” means a straight *worldline*.

Not only must the trajectory of a free stone be straight in an inertial frame, but the stone must also move with constant velocity as measured in that frame. Figure 2 shows a plot of the position of the stone (horizontal axis) as time passes (vertical axis). The straight line in spacetime traced out by the moving stone is a **worldline** (Chapter 1, page 7). Constant velocity results in a *straight worldline.* “Follow a straight worldline in an inertial frame!” is the command by which spacetime grips mass, telling it how to move. No instruction could be simpler.

During the trip between Event 1 and Event 2 in Figure 2, the stone’s wristwatch ticks off intermediate events along the worldline (event-points on the straight worldline of Figure 2).

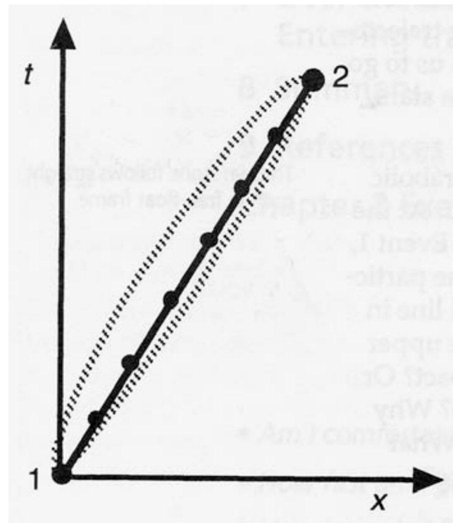

FIGURE 2 Spacetime diagram of the stone's worldline in the inertial frame that rises and falls with the stone (right diagram of Figure 1). This worldline is straight between launch (Event 1) and impact (Event 2). Intermediate clock ticks are shown as event-points along the worldline. Curved dashed lines between Events 1 and 2 represent alternative worldlines of smaller aging, alternative worldlines that the stone does not take.

How is the straight worldline different from all other possible worldlines that connect Event 1 and Event 2 (dashed lines in Figure 2)? We know the answer to that question too, from the Principle of Maximal Aging in flat spacetime: The actual worldline has the longest total wristwatch time of all possible worldlines between fixed events in an inertial frame. The free stone progresses uniformly from one event to the other, without jerks, jolts, or any other kind of acceleration, thereby recording the longest possible time on its wristwatch between these two end-events. In contrast, a frantic traveler starting at the same Event 1 races at near-light speed to the Moon, then streaks back in time for obligatory Event 2. The frantic traveler's wristwatch reads less elapsed time between Events 1 and 2 than does the wristwatch of the relaxed stone. The essential lesson of the twin "paradox" (Section 4 of Chapter 1) is that the *natural* motion between two events has *maximum* wristwatch time in an inertial frame.

Natural motion has maximum wristwatch time: maximum *aging*.

No frantic trip as far as the Moon is necessary to demonstrate the basic principle: *any* deviation whatsoever from the straight worldline, no matter how small, leads to a shorter elapsed wristwatch time. The stone's wristwatch, accurate beyond all human timepieces, detects this difference and traces out the worldline of maximum wristwatch time. Wonder of wonders, the stone sniffs out and follows the worldline of maximum proper time without any wristwatch at all! How? Simply by going straight in inertial spacetime. And

Wristwatch time is an invariant, the same for all observers.

BOX 1: More About the Black Hole

The term "black hole" was adopted in 1967 (by John Wheeler), but the concept is old. As early as 1783, John Michell argued that light must "be attracted in the same manner as all other bodies" and therefore, if the attracting center is sufficiently massive and sufficiently compact, "all light emitted from such a body would be made to return toward it." Pierre-Simon Laplace came to the same conclusion in 1795, apparently independently, and went on to reason that "it is therefore possible that the greatest luminous bodies in the universe are on this very account invisible."

Michell and Laplace used Isaac Newton's "action-at-a-distance" theory of gravitation in analyzing escape of light from or its capture by an already existing compact object. (See "BOX 4. Newton Predicts the Black Hole?".) But is such a static compact object possible? In 1939, J. Robert Oppenheimer and Hartland Snyder published the first detailed treatment of gravitational collapse within the framework of Einstein's theory of gravitation. Their paper predicts the central features of nonspinning black holes described in this book.

Ongoing theoretical study has shown that the black hole is the result of natural physical processes. A nonsymmetric collapsing system is not necessarily blown apart by its instabilities but can quickly—in seconds!—radiate away its turbulence as gravitational waves and settle down into a stable structure. In its final form a black hole has three properties and three properties only: mass, charge, and angular momentum. No other property remains of anything that combined to form the black hole, from pins to palaces. This absence of all detail beyond these three properties has led to the saying (also by Wheeler) "The black hole has no hair."

An uncharged nonspinning black hole is completely described by the Schwarzschild metric, derived in 1915 by Karl Schwarzschild from Einstein's equations for general relativity. The energy of a nonspinning black hole is not available for use outside its horizon. For this reason, a nonspinning black hole is called a "dead black hole."

In contrast to the spinlessness of a dead black hole, the typical black hole, like the typical star, has a spin, sometimes a great spin. The energy stored in this spin, moreover, is available for doing work: for driving jets of matter and for propelling a spaceship. In consequence, the spinning black hole deserves and receives the name "live black hole." It has an angular momentum of its own.

A spinning black hole—or any spinning mass—drags around with it spacetime in its vicinity (see Project 7, The Spinning Black Hole). Near the Earth it is a small effect and has not yet been unambiguously measured directly. Theory predicts that near a rapidly-spinning black hole, the such effects can be large, even irresistible, dragging along nearby spaceships no matter how strong their rockets.

Black holes in Nature appear to be divided roughly into two groups: Some have several times the mass of Sun. Others are monsters with millions—even billions—of times the mass of our Sun and are typically located near the centers of galaxies, where a concentration of matter helps them grow.

Roy P. Kerr derived a metric for an uncharged spinning black hole in 1963, followed in 1967 by a more convenient global coordinate system devised by Robert H. Boyer and Richard W. Lindquist. In 1965 Ezra Theodore Newman and others solved the Einstein equations for the spacetime geometry around an *electrically charged* spinning black hole. Subsequent research has proved that around a steady-state black hole of specified mass, charge, and angular momentum, Kerr-Newman geometry is the *only* solution to Einstein's field equations.

the wristwatch time is *proper time*, an invariant, with the same value as observed in any frame, and verified by direct observation by anyone.

Einstein: Gravity is fictitious!

Figures 1 and 2 reflect the fact that, for slow speed and weak gravitational interaction, Newton's mechanics correctly describes the contrast between a straight worldline in spacetime and a curved path in space. So what's new about relativity? Einstein says that you can do away entirely with Newton's gravitational force. Gravity is fictitious in the sense that you can eliminate it locally by dropping into a free-fall frame.

2 ■ PRINCIPLE OF MAXIMAL AGING

Try all possible worldlines across two adjoining patches.
The free stone chooses the worldline of maximal aging.

How to apply the Principle of Maximal Aging to curved spacetime?

The Principle of Maximal Aging was introduced in special relativity, where it applies to entire worldlines in flat spacetime. In contrast, curved spacetime is effectively flat only on local patches, which must be tiled together for a global span. We change the Principle of Maximal Aging as little as possible in curved spacetime by applying it across each pair of adjoining flat patches.

> ***Principle of Maximal Aging (curved spacetime):*** The worldline that a free stone takes across two adjoining spacetime patches is the worldline for which the total wristwatch time (the total aging) across the two patches is a maximum.

General relativity stitches together "quilt squares" of local patches.

The Principle of Maximal Aging determines the worldline across *every pair* of adjoining patches in curved spacetime. Curved spacetime can be completely tiled with adjoining patches, so the stone learns—step by step—how to move globally in curved spacetime. General relativity stitches together local quilt patches into a full quilt that spans global regions of spacetime.

Suppose that the stone rebels; let it disobey the command issued by spacetime to follow the worldline of maximal aging across adjoining patches. Or more realistically, think of an external experimenter who grasps the stone and forces it to move along a worldline that it would not freely follow. This rebellion, this deviation from the natural, is partial: the stone is present at the two obligatory events at the leading edge of the first patch and at the final edge of the second patch. However, the rebel stone does not keep its appointments with the intermediate events along the standard, the natural, the actual worldline of the free stone. Perhaps it moves slower than normal between adjacent points on parts of the spatial path and faster than normal on other parts. Or perhaps it wanders off the spatial path entirely, taking some other trajectory. Nevertheless, its wristwatch continues conscientiously to tick off wristwatch time—accumulated aging—along this alternative worldline. In due course the stone arrives at the obligatory final event. The stone's penalty for its errant behavior? A mild punishment! At the end-event on the second patch the stone's wristwatch will read less time than it would if it had obeyed the command of spacetime. The errant stone's aging for this segment of worldline will not be maximal among all possible worldlines between initial and final events on the adjoining patches.

Pick the actual worldline: the one with maximal wristwatch time.

The disobedient stone shows us how to predict—simply, accurately, powerfully—the worldline of any test object moving freely across adjoining patches of spacetime, no matter whether the spacetime region is curved or flat. The recipe could hardly be simpler: "Behave like a large number of rebellious stones!" Each rebellious stone follows a different worldline from initial event to final event. Compute the aging along each alternative worldline—the sum of incremental wristwatch times between each pair of adjacent events along the

BOX 2: THE PRINCIPLE OF MAXIMAL AGING IS OK IN CURVED SPACETIME

We know that the stone follows the Principle of Maximal Aging in flat spacetime (Chapter 1). But why does it follow the Principle of Maximal Aging in curved spacetime? Because the stone thinks it is always in flat spacetime! Picture the stone from instant to instant, always in the center of a flat patch, with Cartesian coordinates making it a frame. In this frame the stone does the most natural thing possible: it moves straight in space at constant speed—that is, on a straight worldline. In Newton's words, the stone "perseveres in its state of being at rest or of moving uniformly straight forward . . ." What could be simpler?

Are we satisfied with this description of the stone's motion: "go straight in the local free-fall frame"? No, we want more; we want to find the *global* motion of the stone entirely around Earth or black hole. To start toward that goal we track the stone across any two adjoining local frames each of which is small enough to be effectively flat. We vary the time of crossing between frames to maximize the total time across both of them measured on the stone's wristwatch. The result is a quantity that has the same value across each adjoining frame—a constant of the stone's motion. These two adjoining frames could be *anywhere* outside the horizon of a black hole. Therefore the resulting expression is correct anywhere in that space and for all time.

How do we know that the constant of motion we identified is the energy of the stone and not some other quantity? Because it reduces to the expression for energy in flat spacetime when we let the mass M of the black hole goes to zero.

Is there any circumstance in which the Principle of Maximal Aging does not work to find the motion of the stone? Yes, when the space curvature changes significantly from one part of the stone to another. Then there is no patch small enough so that the stone is effectively in a flat region of spacetime. Where in the Universe will that happen? At the singularity in a black hole, where space curvature increases without limit.

candidate worldline. Among all these candidates, select the worldline with maximal aging. The maximal-aging worldline is the one taken by the real stone, the stone moving freely between fixed initial and final events.

?

Your theory is fundamental and interesting—and useless! How can we predict the motion of the stone if we need to know from the beginning the "fixed" final event on the worldline—the place and time of impact? The location of that final event is just what the laws of motion are supposed to TELL us: Given the launch point and the initial velocity, where will the projectile impact? Usually we don't even care WHEN it reaches that point. For such an analysis, your prescription is useless.

! ¶

No, not useless. Think of trapshooting (or skeet shooting), a sport in which we fire buckshot pellets at a ceramic target ("clay pigeon") launched by a spring. We know the trajectory of the clay pigeon in advance, or we can predict this trajectory. Hitting the clay pigeon requires taking account of both location and time of impact between shot and clay pigeon. The tight packet of shotgun pellets must cross the trajectory of the clay pigeon WHEN the clay pigeon is at that particular point in space. In brief, fix both the space and time location of a final impact *event*. The initial launch event is the firing of the shotgun. Think of a computer program that selects spacetime events of launch and impact, tries out various alternative worldlines between the two events, selects the worldline of maximal aging,

BOX 2. What Then Is Time?

What then is time? If no one asks me, I know what it is. If I wish to explain it to him who asks me, I do not know.

The world was made, not in time, but simultaneously with time. There was no time before the world.

—St. Augustine (354–430 C.E.)

Time takes all and gives all.

—Giordano Bruno (1548–1600 C.E.)

Everything fears Time, but Time fears the Pyramids.

—Anonymous

Philosophy is perfectly right in saying that life must be understood backward. But then one forgets the other clause—that it must be lived forward.

—Søren Kierkegaard

As if you could kill time without injuring eternity.

Time is but the stream I go a-fishing in.

—Henry David Throeau

I do not define time, space, place and motion, [because they are] well known to all.

—Isaac Newton

Time is defined so that motion looks simple.

—Misner, Thorne, and Wheeler

Nothing puzzles me more than time and space; and yet nothing troubles me less, as I never think about them.

—Charles Lamb

Either this man is dead or my watch has stopped.

—Groucho Marx

“What time is it, Casey?”

“You mean right now?”

—Casey Stengel

It’s good to reach 100, because very few people die after 100.

—George Burns

Time is Nature’s way to keep everything from happening all at once.

—Graffito, men’s room, Pecan St. Cafe, Austin, Texas

What time does this place get to New York?

—Barbara Stanwyck, during trans-Atlantic crossing on the steamship *Queen Mary*

and specifies for us the aiming direction (for a given muzzle velocity) to achieve a hit in terms of the specified events of launch and impact. In some cases this procedure can be more useful than the common analysis that starts from initial conditions and predicts subsequent motion. However, one can also do it your way: The following section uses the Principle of Maximal Aging to derive the expression for energy in curved Schwarzschild geometry. This result helps to carry out the more conventional analysis (“predict subsequent motion from data on initial position and velocity”).

3 ■ ENERGY IN SCHWARZSCHILD MAP COORDINATES

Maximal Aging derives energy as a constant of motion.

Energy: constant of motion

This section reveals a new expression for the energy of a free stone, in particular a constant of motion for a stone falling radially toward a nonspinning black hole. In Chapter 1 (page 10), we derived the expression for energy in flat spacetime. Our present derivation extends this analysis to curved spacetime near a nonspinning black hole. The leap forward is not as great as you might think: Instead of a global flat coordinate system, our new analysis uses two flat patches stacked radially on top of one another. Figure 3 shows the space part of these two patches: higher Patch H and lower Patch L. The time part of each (space*time*) patch is the lapse of map time it takes for the plunging stone to cross that patch. As it enters Patch H the stone emits a flash at event labeled P. It emits a second flash (event K) at the boundary between Patches H and L. As it leaves Patch L, the stone emits a final flash at Q.

Find maximal aging: find natural motion.

Our goal is to divide up the stone's total fixed *map* time between the two patches so as to maximize the total *wristwatch* time (aging) across both patches.

?

What is going on? Who cares how much time the stone spends on upper Patch H or on lower Patch L? What does map time have to do with wristwatch time, anyway? Don't throw a lot of equations at me before explaining your goal and the strategy you use to reach this goal!

!

Good advice. Here is the big picture as previewed in Interlude 3, The Patch: First, there are two different times in the analysis, which need to be distinguished: Schwarzschild map time and stone wristwatch time. Second, to satisfy the Principle of Maximal Aging we want the stone's wristwatch to read the maximum total elapsed time as it crosses the two patches. We watch the stone enter at the top of the higher Patch H and exit from the bottom of lower Patch L. We choose to fix these entrance and exit events in both space and time in Schwarzschild map coordinates.

Now, we know that a clock at rest runs faster at a higher altitude $\bar{r}_{\mathrm{H}}$ than at a lower altitude $\bar{r}_{\mathrm{L}}$ (Chapter 2, page 00), where the bar indicates the average radius of each patch. So a possible strategy for the stone is simply to stop on higher Patch H and let its wristwatch accumulate as much time as possible. But if the stone does that, it must zip across lower Patch L at high speed in order to reach the fixed exit event at the required (map) time. And that is the problem: We know that a super-fast wristwatch runs slow. While moving at high speed across lower Patch L the stone's slowed wristwatch will lose some of the extra time it built up while at rest in upper Patch H.

In practice the stone moves so as to compromise the two effects: Move slower in higher Patch H to in order to spend extra time at the higher radius where its wristwatch runs fast. Move faster in order to spend less time on lower Patch L where its wristwatch runs slower—but not so fast that speed-related clock slowing cancels the extra time built up on higher Patch H. Choose actual time of the intermediate event—Event K in Figure 3—to maximize the total wristwatch time across the two patches, thus satisfying the Principle of Maximal Aging.

We find the maximum aging by varying the map time of intermediate event K while keeping fixed all other space and time coordinates of events P, K, and Q. Maximum aging leads to an expression for a quantity that remains constant as the stone falls. That constant of motion is the *energy* of the stone.

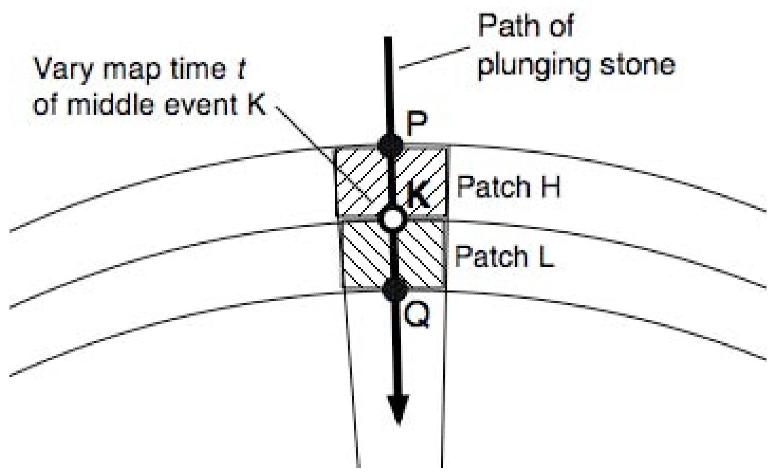

FIGURE 3 Deriving the expression for energy in Schwarzschild geometry using the Principle of Maximal Aging. The plunging stone crosses higher Patch H, then lower Patch L, emitting flashes at events P, K, and Q. Vary the relative times of transit across Patches H and L by varying the time of intermediate event K. Use this variation to maximize the total aging across both patches between fixed events P and Q. The result is an expression for energy as a constant of motion.

The Schwarzschild metric for a radially plunging stone ($d\phi = 0$) tells us the relation between advance $d\tau$ of its wristwatch time and change of map time and radius: (#Schwarz)

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - \dfrac{2M}{r}} \qquad \text{(radial plunge)} \qquad (1)$$

Approximating the Schwarzschild metric

The following analysis examines the wristwatch time and Schwarzschild map time separations between events P, K, and Q. For simplicity, replace the differentials $d\tau$ and dt with symbols τ and t, respectively. We make this replacement to facilitate the maximization of wristwatch time below. Strictly speaking, replacing coordinate differentials by finite quantities is illegal in curved spacetime. But it is okay on each single patch, defined to be a region small enough that curvature effects are negligible. At the end we will return to strictly correct differential notation.

Let T represent the fixed total Schwarzschild map time for the stone to cross both patches and t represent the map time for the stone to cross Patch H. Then the stone takes map time $T - t$ to cross Patch L. In the following we vary time t to maximize the stone's total wristwatch time across both patches.

The symbol $\bar{r}_{\rm H}$ represents an average radius of higher Patch H and $\bar{r}_{\rm L}$ an average radius of lower Patch L. (The *kind* of average does not matter because ultimately we go to the differential limit of narrow radial patch dimension, squeezing every average to the resulting single radial coordinate.)

Wristwatch time (aging) across each patch.

As long as Patches H and L and transit times t and $T - t$ are very small, we can use approximate forms of the Schwarzschild metric (1). We will also be interested only in the parts of the metric that contain the variable t, because we will be taking a derivative with respect to this time. The stone's wristwatch times $\tau_{\rm H}$ and $\tau_{\rm L}$ (aging) while passing across higher and lower patches can then be written: (#tauA)

$$\tau_{\rm H} = \left[\left(1 - \frac{2M}{\bar{r}_{\rm H}}\right) t^2 + (\text{terms without } t)\right]^{1/2} \tag{2}$$

and (#tauB)

$$\tau_{\rm L} = \left[\left(1 - \frac{2M}{\bar{r}_{\rm L}}\right) (T-t)^2 + (\text{terms without } t)\right]^{1/2} \tag{3}$$

To prepare for the derivative that leads to maximal aging, take the derivative of $\tau_{\rm H}$ with respect to t: (#dtauAdt)

$$\frac{d\tau_{\rm H}}{dt} = \frac{\left(1 - \frac{2M}{\bar{r}_{\rm H}}\right) t}{\left[\left(1 - \frac{2M}{\bar{r}_{\rm H}}\right) t^2 + (\text{terms without } t)\right]^{1/2}} = \left(1 - \frac{2M}{\bar{r}_{\rm H}}\right) \frac{t}{\tau_{\rm H}} \tag{4}$$

The corresponding expression for $d\tau_{\rm L}/dt$ is: (#dtauBdt)

$$\frac{d\tau_{\rm L}}{dt} = -\frac{\left(1 - \frac{2M}{\bar{r}_{\rm L}}\right) (T-t)}{\left[\left(1 - \frac{2M}{\bar{r}_{\rm L}}\right) (T-t)^2 + (\text{terms without } t)\right]^{1/2}} = -\left(1 - \frac{2M}{\bar{r}_{\rm L}}\right) \frac{T-t}{\tau_{\rm L}} \tag{5}$$

Add the two wristwatch times to obtain the total wristwatch time between first and last events P and Q: (#TotalTau)

$$\tau_{\rm total} = \tau_{\rm H} + \tau_{\rm L} \tag{6}$$

Maximize total aging.

The Principle of Maximal Aging says that the natural motion yields a maximum for the total wristwatch time $\tau_{\rm total}$ (total aging) across the two patches. To find this maximum, take the derivative of both sides of (6) with respect to t, substitute from (4) and (5). and set the result equal to zero: (#dtautotal)

$$\frac{d\tau_{\rm total}}{dt} = \frac{d\tau_{\rm H}}{dt} + \frac{d\tau_{\rm L}}{dt} = \left(1 - \frac{2M}{\bar{r}_{\rm H}}\right) \frac{t}{\tau_{\rm H}} - \left(1 - \frac{2M}{\bar{r}_{\rm L}}\right) \frac{(T-t)}{\tau_{\rm L}} = 0 \tag{7}$$

From the last equality in (7), (#EnergyA)

$$\left(1 - \frac{2M}{\bar{r}_{\rm H}}\right) \frac{t}{\tau_{\rm H}} = \left(1 - \frac{2M}{\bar{r}_{\rm L}}\right) \frac{(T-t)}{\tau_{\rm L}} \tag{8}$$

Define $t_{\mathrm{H}} \equiv t$ and $t_{\mathrm{L}} \equiv T - t$, so (8) becomes (#EnergyB)

$$\left(1 - \frac{2M}{\bar{r}_{\mathrm{H}}}\right)\frac{t_{\mathrm{H}}}{\tau_{\mathrm{H}}} = \left(1 - \frac{2M}{\bar{r}_{\mathrm{L}}}\right)\frac{t_{\mathrm{L}}}{\tau_{\mathrm{L}}} \qquad (9)$$

Energy in Schwarzschild map coordinates

The expression on the left side of (9) depends only on the parameters of the higher Patch H; the expression on the right side depends only on the parameters of the lower Patch L. Hence the value of either side of this equation must be independent of which adjoining pair of segments we choose to look at: equation (9) displays a quantity that has the same value on *every* segment of the path. We have found the expression for a quantity that is a constant of motion. Contract the patches to their differential limits. The result is an expression that we can identify as the stone's *energy*:(#FinalEnergy)

$$\boxed{\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}} \qquad (10)$$

Energy takes special relativity form far from black hole.

Equation (10) in differential form gives E/m at the single radius r, so the average bar over the symbol can be omitted. Identification with energy E follows by noting that when the mass M of the center of attraction goes to zero—or for locations far from the center of attraction where $2M/r \ll 1$—the expression reduces to that for energy in special relativity, $E/m = dt/d\tau$ (Chapter 1, page 13).

Rather than focus on E alone, we use the dimensionless ratio E/m. Why? For two reasons: (a) We recognize that stones of different mass m follow the same worldline through spacetime. What counts for motion is neither the mass of the plunging stone by itself nor its energy by itself but only the ratio of the two, the energy per unit mass. (b) The ratio E/m has no units provided we express E and m in the same unit, a unit that we may choose according to convenience and the experiment being described. Both numerator and denominator in E/m may be expressed in kilograms or joules or the mass of the proton or million electron-volts, and so on.

Energy expression also correct for non-radial motion of stone.

This derivation employs only the time part of the metric. It makes no difference in the outcome—expression (10) for energy—whether dr and $d\phi$ are zero or not. This has an immediate practical consequence, namely that the same expression for energy is as valid for a stone moving around a spherically symmetric center of attraction as for one plunging radially inward or coasting radially outward. We will use this generality of (10) for predicting orbits in Chapter 4.

4 ■ MAP ENERGY VS. MEASURABLE ENERGY

Map energy as a unicorn: a mythical beast

The expression on the right side of equation (10) is a unicorn: a mythical beast. What does this mean: a mythical beast? It means that nobody

Map energy E/m is a unicorn.

measures directly either the radius r or the differential time lapse dt. Why not? Because these are *Schwarzschild map coordinates*, entries in the mapmaker's spreadsheet or accounting form, not coordinates that any actual observer measures. If this is so, why did we bother to derive expression (10) in the first place? Because it has one primary virtue: it is valid anywhere near a nonspinning black hole. In contrast, individual viewers are local; they measure local coordinates, such as shell coordinates on a shell

Shell observer's expression for energy?

What will a *real* experimenter observe as as a stone plunges past him? In what kind of experiment will energy (10) be of practical use? How does E relate to measurements? Think of a shell observer: How much energy can he extract from a falling stone? Call the energy that the shell observer measures for the stone the **shell energy** E_{shell}. What is the equation for E_{shell}?

To find the expression for shell energy from (10), we need to convert dt to dt_{shell}. (It's OK to leave r in the resulting expression, since the numerical value of r is stamped on every shell during construction.) Our standard conversion between map and shell time increments is: (#inertialshellenergy)

$$dt_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} dt \qquad (11)$$

With substitution (13) of time conversion, equation (10) takes the form: (#ShellEnergy)

$$\boxed{\frac{E}{m} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{dt_{\text{shell}}}{d\tau} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{E_{\text{shell}}}{m} = \left(\frac{1 - \dfrac{2M}{r}}{1 - v_{\text{shell}}^2}\right)^{1/2}} \qquad (12)$$

The final steps in this equation make use of the expression for energy in flat spacetime from page 00 of Chapter 1. Using shell coordinates, the flat-spacetime expression for energy is (#ShellConversion)

$$\frac{E_{\text{shell}}}{m} = \frac{dt_{\text{shell}}}{d\tau} = \frac{1}{\left(1 - v_{\text{shell}}^2\right)^{1/2}} \qquad (13)$$

This is permitted because the shell patch is, by definition, locally inertial. In an inertial frame the rest energy has the value one. (#restshellenergy)

$$E_{\text{shell rest}} = m \qquad \text{or} \qquad E_{\text{shell rest conv}} = mc^2 \qquad (14)$$

Equation (12) allows us to find the value of E/m for any plunging stone by measuring its shell velocity v_{shell} and reading the r-value stamped on the shell.

"Energy at infinity"

Note that for very large radius, (#EovermAtInfinity)

$$\frac{E}{m} \to \frac{E_{\text{shell}}}{m} \qquad (r \gg 2M) \qquad (15)$$

This is not surprising, since spacetime is flat far from the black hole. Indeed, far from the black hole E/m is *not* a unicorn, a mythical beast, because there the coordinate time t is measured directly. For this reason energy E in (10) and (refShellEnergy) is sometimes called "**energy at infinity**."

From equation (12) we can illustrate the energy and the shell energy for various initial conditions: (#rainEnergy)

$$\frac{E}{m} = 1 \qquad \text{(dropped from rest } r \gg 2M\text{)} \tag{16}$$

(#dripenergy)

$$\frac{E}{m} = \left(1 - \frac{2M}{r_0}\right)^{1/2} \qquad \text{(released from rest at } r_0\text{)} \tag{17}$$

(#hailEnergy)

$$\frac{E}{m} = \frac{1}{(1 - v_{\mathrm{far}}^2)^{1/2}} \qquad \text{(hurled inward at } v_{\mathrm{far}} \text{ from a great distance)} \tag{18}$$

In using these equations, the right hand sides remain constant as the stone plunges inward.

Raindrop: dropped from rest at infinity.

Equation (16) is a case we will use extensively from now on. Our name for a stone released from rest at $r_0 \to \infty$ is **raindrop**, because rain falls from a great height. The energy of a raindrop is m, its rest energy at infinity. Its shell energy comes from letting $r_0 \to \infty$ in (35): (#RaindropShellEnergy)

$$\frac{E_{\mathrm{shell}}}{m} = \left(1 - \frac{2M}{r}\right)^{-1/2} \qquad \text{(raindrop)} \tag{19}$$

The shell speed of a raindrop at r comes from the same limiting case of (37): (#RainShellSpeed)

$$\frac{dr_{\mathrm{shell}}}{dt_{\mathrm{shell}}} = v_{\mathrm{shell}} = -\left(\frac{2M}{r}\right)^{1/2} \qquad \text{(raindrop shell speed)} \tag{20}$$

Combine (16) with (10) and use the Schwarzschild metric to find an expression for the map speed of a raindrop (see the exercises): (#drdt)

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2} \qquad \text{(raindrop map speed)} \tag{21}$$

Equation(52) shows an apparently outrageous result, namely that as the raindrop approaches the horizon its Schwarzschild map speed decreases, and the stone coasts to zero map speed at the horizon. Repeated use of the word "map" reminds us that map speeds are simply spreadsheet entries for the Schwarzschild mapmaker and do not correspond to direct measurements by any local observer. Nothing could demonstrate more clearly the radical difference between map entries and direct observation.

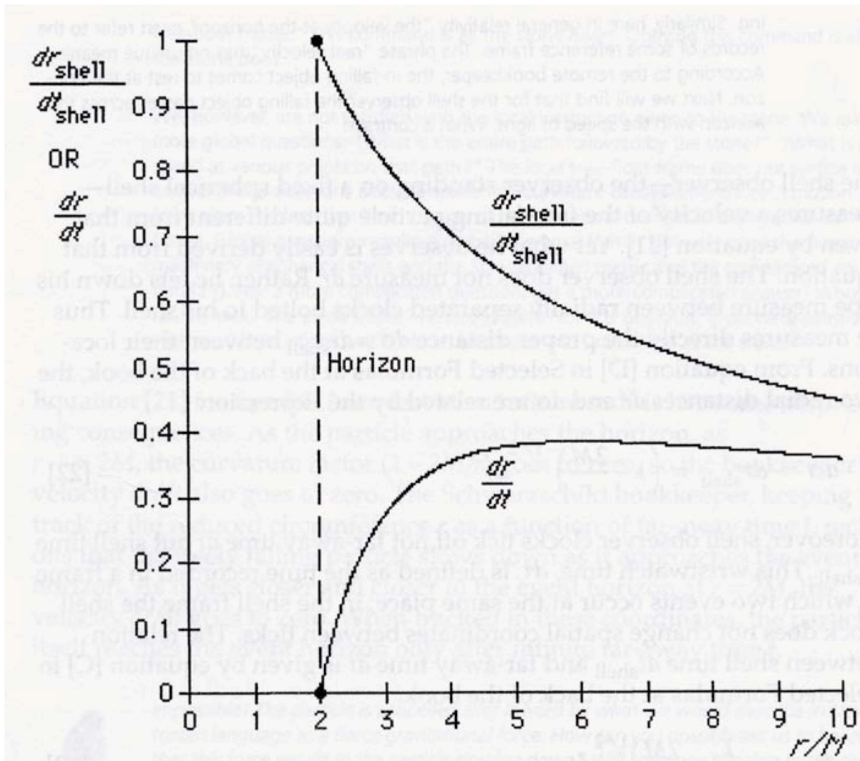

FIGURE 4 Computer plot of the two values of speed for a raindrop. The raindrop falling radially from rest at infinity has speed $|dr_{\rm shell}/dt_{\rm shell}|$ as measured by observers on shells through which the stone plunges and speed $|dr/dt|$ as derived from the records of the Schwarzschild mapmaker. At the horizon, the shell speed rises to the speed of light, equation (39), while the map speed drops to zero, equation (36). [CHANGE "OR" TO "AND" IN LABEL OF VERTICAL AXIS AND PUT ABSOLUTE VALUE SIGNS ON VERTICAL AXIS EXPRESSIONS.]

Figure 00 shows plots of both shell and map speeds of the descending raindrop.

SAMPLE PROBLEM 1. The Neutron Star Takes an Aspirin

Neutron Star Gamma has a total mass 1.4 times that of our Sun and a map radius $r_{\rm surface} = 10$ kilometers. An aspirin tablet of mass one-half gram falls from rest at a great distance onto the surface of the neutron star. An advanced civilization converts the kinetic energy of the aspirin tablet into useful energy. Estimate how long this energy will power a 100-watt bulb.

SOLUTION

From the value of our Sun's mass inside the front cover, the mass of the neutron star is $M \approx 2 \times 10^3$ meters. Hence $2M/r_{\rm surface} \approx 2/5$. The total energy E of the aspirin tablet equals its energy at rest far from the neutron star, namely its mass m. From

(12), the shell energy of the aspirin tablet at the surface of the neutron star is (#SurfaceEnergy)

$$\frac{E_{\text{shell}}}{m} = \left(1 - \frac{2M}{r}\right)^{-1/2} \approx 1.3 \tag{22}$$

where $r = r_{\text{surface}}$. The shell *kinetic energy* of the half-gram aspirin tablet is the difference between its shell energy and its rest energy m, so from (14) its shell kinetic energy is 0.15 gram or 1.5×10^{-4} kilogram. Multiply by c^2 to obtain energy in joules. The result is 1.4×10^{13} joules. One watt is one joule/second; a 100-watt bulb consumes 100 joules per second. At that rate, the bulb can burn for 1.4×10^{11} seconds on the kinetic energy of the aspirin tablet. One year is about 3×10^7 seconds. Result: The kinetic energy of the aspirin tablet at the surface of Neutron Star Gamma can light a 100-watt bulb for almost five thousand years.

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5 ■ MASS OF A CENTER OF ATTRACTION MEASURED FROM A GREAT DISTANCE

A new way to measure total energy

How can we understand the conserved "energy at infinity" E when the Newtonian approximation breaks down? We call E an energy, but is it really? Can it be converted to other forms of energy? Can its value even be measured?

Satellite responds to gravity, also creates gravity.

To answer these questions, consider the quantity $E - E_{\text{shell}} = E_{\text{far}} - E_{\text{shell}}$ for a stone, where we equate the energy E of a stone to the energy E_{far} it would have at infinity. But gravity acts in two ways: a satellite of mass m creates its own gravitational field even as it responds to the gravity of a nearby star of mass M_{star}. This suggests a new way to measure gravitational energy. Up until now our satellites have been *stones*, defined as free particles "whose mass warps spacetime too little to be measured" (inside back cover). But now, in order to measure the energy E of the satellite, we pay attention to the combined gravitational effect of the star plus the satellite.

Figure 6 illustrates how the gravitational mass M_{total} of the combined star-plus-satellite system might be measured using the acceleration of a distant test particle so remote that Newtonian attraction supplies an accurate tool for measuring mass. In geometric units Newton's expression for this acceleration is: (#NewtAccel)

$$a = -\frac{M_{\text{total}}}{r^2} \qquad \text{(Newton)} \tag{23}$$

What is M_{total}? In Newtonian mechanics gravitational masses add: (#NewtonGravMass)

$$M_{\text{total}} = M_{\text{star}} + m \qquad \text{(Newton)} \tag{24}$$

where m is the mass of the satellite. Could this also be true in general relativity? No! Equations (13) and (15) show that the mass of the satellite far

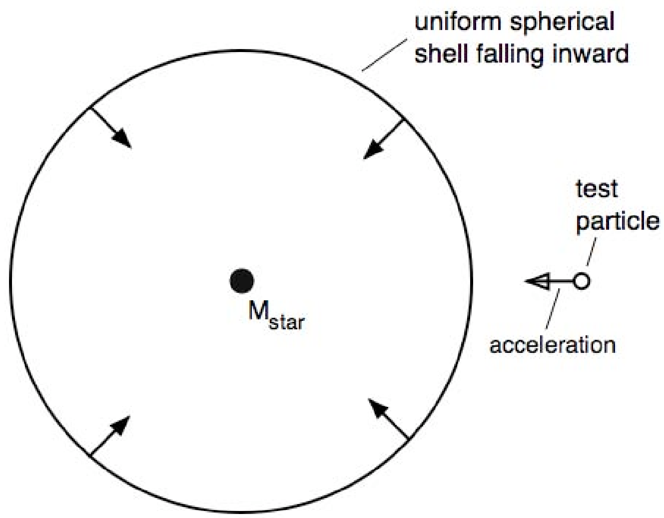

FIGURE 5 Replacing the moving satellite of Figure 6 with an inward-falling uniform sphere that satisfies the condition of Birkhoff's theorem, so that the Schwarzschild metric applies outside the contracting shell.

from a black hole is $E_{\rm shell}$, not m. This suggests that we try $M_{\rm total} = M_{\rm star} + E_{\rm shell}$, where $E_{\rm shell}$ is given by (13). However, this formula implies incorrectly that $M_{\rm total}$ is not a constant: As a stone plunges to the surface of a neutron star, $E_{\rm shell}$ increases as shown by (22). Therefore $M_{\rm total}$ increases. Can this be right? No, and here is why.

Birkhoff's theorem

A mathematical theorem of general relativity due to G. D. Birkhoff in 1923 states that the spacetime outside any spherical distribution of matter and energy is completely described by the Schwarzschild metric with a *constant* gravitational mass $M_{\rm total}$. Figure 5 displays a circumstance in which Birkhoff's theorem applies, so that the gravitational mass detected by the observer external to the shell is constant. In contrast, $M_{\rm star} + E_{\rm shell}$ changes as the satellite/shell plunges inward, so it cannot equal the gravitational mass $M_{\rm total}$.

To make the problem easier, we are going to approximate the moving satellite by an inward-falling uniform sphere that satisfies the condition of Birkhoff's theorem, so that the Schwarzschild metric applies outside this inward-falling shell/satellite (Figure 5).

Unfortunately, Birkhoff's theorem does not tell us how to calculate the value of $M_{\rm total}$, only that it is a constant for a spherical configuration of mass/energy. We need to replace $E_{\rm shell}$ by an energy that does not change as mass moves inward (or outward). Our constant-of-motion energy E is a possible candidate, an energy given by (12): (#SatelliteEnergyFormula)

$$E = \left(1 - \frac{2M}{r}\right)^{1/2} E_{\rm shell} \qquad (25)$$

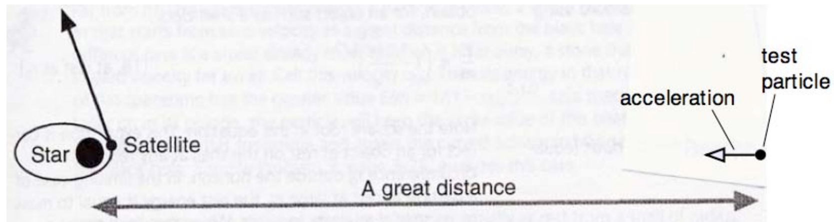

FIGURE 6 Measuring the total mass-energy M_{total} of a central star-satellite system using the acceleration of a test particle at a great distance.

Add E to M_{star} to get M_{tot}.

So instead of the Newtonian expression (24) we have the trial general relativity replacement: (#GRGravMass)

$$M_{\text{total}} = M_{\text{star}} + E \qquad \text{(Einstein)} \tag{26}$$

How do we know that the energy-at-infinity E is the right constant to add to M_{star}? One check is that when the satellite/shell is far from the star ($r \gg 2M_{\text{total}}$ but the remote test particle is still exterior to the shell) then $E \rightarrow E_{\text{shell}}$. For a slowly moving satellite/shell, $E_{\text{shell}} \rightarrow m$, and we recover Newton's formula (24), exactly as we should. And when the satellite/shell falls inward so that $E_{\text{shell}} > m$, equation (26) remains valid, because $E(\approx m)$ does not change.

If (26) is correct, then general relativity merely replaces Newton's m with the conserved value E of the satellite/shell. The mass of a star or black hole grows by the energy E of a stone or collapsing shell that falls into it. The energy of the stone has been converted into gravitational mass.

?

You checked equation (26) only in the Newtonian limit, where the remote shell is at rest or falls inward with small kinetic energy. Is (26) valid for all values of E? Suppose that the collapsing shell in Figure 5 is hurled inward (or outward) at relativistic speed. In this case does total E still simply add to M_{star} to give total mass M_{total} for the still more distant observer?

!

Yes it does, but we have not displayed the proof, which requires solution of Einstein's equations. Let a massive star collapse, then explode into a supernova. If this process is spherically symmetric, then a distant observer will detect no change in gravitational attraction in spite of the radical conversions among different forms of energy. Actually, the distant observer has no way of knowing about these transformations before the outward-blasting shell of radiation and neutrinos passes her. When that happens she will detect a gravitational decline in the mass of the central attractor because some of its original energy has been carried to a radius greater than hers.

Gravity waves carry off energy.

Is the Birkhoff restriction to spherical symmetry important? It can be: A satellite orbiting or falling into a star or black hole will emit gravitational waves that carry away some energy, decreasing M_{total}. Project 9, Gravitational Waves, notes that a spherical distribution cannot emit gravitational waves, no matter how that spherical distribution moves in and out. As a result, equation (26) is okay to use only when the emitted gravitational wave energy is very much less than M_{total}. When that condition is met, the cases shown in Figures 6 and 5 are observationally indistinguishable.

Measuring E from a distance.

We can, in principle, use (26) to measure the energy E of *anything* circulating about, plunging into, launching itself away from, or otherwise interacting with a center of attraction—as long as gravitational wave emission is not a factor and we are sufficiently far from the objects. Simply use Newtonian mechanics to carry out the measurement depicted in Figure 6, first with the satellite absent, second with the satellite/shell in place near the star. Subtract the second value from the first for the acceleration (23) and use (26) to determine the value of $E = M_{\text{total}} - M_{\text{star}}$.

6 ■ FALLING RADIALLY INWARD: DRIPS, RAINDROPS, AND HAILSTONES

Let go or hurl inward.

Drip: dropped from rest at r_0.

The fact that E/m in (10) is a constant of motion for a free particle yields great simplification in describing the motion of a radially plunging stone. As an example, think of a stone released from rest at initial map radius r_0 near a nonspinning black hole. We call this falling object a **drip** because it drips from rest, as from a leaky faucet. The energy of a drip is given by (**??**), from which its shell energy can be derived using (12): (#DripShellEnergy)

$$\frac{E_{\text{shell}}}{m} = \left(1 - \frac{2M}{r_0}\right)^{1/2}\left(1 - \frac{2M}{r}\right)^{-1/2} \qquad \text{(drip)} \qquad (35)$$

The map speed $|dr/dt|$ of the drip is already given by (52), derived on the way to finding initial gravitational acceleration on the shell. (#drdtRepeat)

$$\left|\frac{dr}{dt}\right| = \left(1 - \frac{2M}{r_0}\right)^{-1/2}\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r} - \frac{2M}{r_0}\right)^{1/2} \qquad \text{(drip speed)} \quad (36)$$

Radial shell speed follows from equation (**??**), so that (36) gives us: (#DripShellSpeed)

$$|v_{\text{shell}}| = \left(1 - \frac{2M}{r}\right)^{-1}\left|\frac{dr}{dt}\right| = \left(1 - \frac{2M}{r_0}\right)^{-1/2}\left(\frac{2M}{r} - \frac{2M}{r_0}\right)^{1/2} \qquad \text{(drip)} \quad (37)$$

Raindrop: dropped from rest at infinity.

Our name for a stone released from rest at $r_0 \to \infty$ is **raindrop**, because rain falls from a great height. (The raindrop is a special case of a drip.) The

BOX 3. Baked on the Shell?

As you stand on a spherical shell close to the horizon of a black hole, you will be crushed by an unsupportable local gravitational acceleration directed downward toward the center. If that is not enough, you will also be enveloped by an electromagnetic radiation field. William G. Unruh used quantum field theory to show that the temperature *T* of this radiation field in degrees Kelvin is given by the equation (#eq:42)

$$T = \frac{h g_{\mathrm{conv}}}{4\pi^2 k_{\mathrm{B}} c} \tag{27}$$

Here g_{conv} is the local acceleration of gravity in the conventional units meters/second^2, h is Planck's constant, c is the speed of light, and k_{B} is the so-called **Boltzmann's constant,** which has the value 1.381×10^{-23} kilogram-meters^2/(second^{2}degree Kelvin). The quantity $k_{\mathrm{B}}T$ has the unit joules and gives an average value for the thermal energy this field can provide to local processes. (The same radiation field surrounds you when you accelerate at the rate g_{conv} in flat spacetime.)

We are deriving an expression for the local gravitational acceleration on a shell at radius r. This acceleration, expressed in the geometric unit meter^{-1} is given in (59): (#eq:43)

$$g_{\mathrm{shell}} = \frac{g_{\mathrm{conv}}}{c^2} = -\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2} \tag{28}$$

Substitute g_{conv} from (28) into (27) to obtain (#eq:44)

$$T = \frac{hc}{4\pi^2 k_{\mathrm{B}}}\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2} \tag{29}$$

where M is in meters. This temperature increases without limit as you approach the horizon at $r = 2M$. Therefore one would expect the radiation field near the horizon to shine brighter than any star when viewed by a distant observer. Why doesn't this happen? In a muted way it does happen. Remember that radiation is gravitationally red-shifted as it moves away from any center of gravitational attraction. From equation [C] in Selected Formulas at the end of this book we can show that every frequency is red-shifted by the factor $(1 - 2M/r)^{1/2}$, which cancels the corresponding factor in (29). Let $r \to 2M$ in the resulting equation. The distant viewer sees the radiation temperature (#eq:45)

$$T_H = \frac{hc}{16\pi^2 k_{\mathrm{B}} M} \tag{30}$$

where M is in meters. The temperature T_{H} is called the **Hawking temperature** and characterizes the Hawking radiation from a black hole, described in Box 1 of Chapter 2 (page 6). Notice that this temperature *increases* as the mass M of the black hole *decreases*. For a black hole whose mass is a few times that of our Sun, this temperature is extremely low, so from a distance such a black hole really looks *almost* black.

The radiation field described by equations (27) through (29), although perfectly normal, leads to strange conclusions. Perhaps the strangest of all is that this radiation field is entirely undetected by a free-fall plunging observer who passes the shell at radius r. The plunging traveler observes no such radiation field, while for the shell observer at the same radius the radiation is a surrounding presence. This apparent paradox cannot be resolved using the classical theory developed in this book; see Kip Thorne's *Black Holes and Time Warps: Einstein's Outrageous Legacy*, page 444.

How realistic is the danger of being baked on a shell near the horizon of a black hole? In answer, compute the local acceleration of gravity for a shell on which the radiation field reaches a temperature equal to the freezing point of water, 273 degrees Kelvin. From (27) you can show that $g_{\mathrm{conv}} = 6.7 \times 10^{22}$ meters/second^2, or almost 10^{22} times the acceleration of gravity on Earth's surface. Evidently we will be crushed by gravity long before we are baked by radiation!

energy of a raindrop is m, its rest energy at infinity. Its shell energy comes from letting $r_0 \to \infty$ in (35): (#RaindropShellEnergy)

$$\frac{E_{\mathrm{shell}}}{m} = \left(1 - \frac{2M}{r}\right)^{-1/2} \qquad \text{(raindrop)} \tag{38}$$

The shell speed of a raindrop at r comes from the same limiting case of (37): (#RainShellSpeed)

$$|v_{\mathrm{shell}}| = \left(\frac{2M}{r}\right)^{1/2} \qquad \text{(raindrop speed)} \tag{39}$$

SAMPLE PROBLEM 3. Examples of Gravitational Acceleration

1. On a shell at $r/M = 4$ near a black hole, the initial gravitational acceleration from rest is how many times that predicted by Newton?
2. On a shell at $r/M = 2.1$ near a black hole, the initial gravitational acceleration is how many times that predicted by Newton?
3. What is the minimum value of r/M so that, at or outside of that radius, Newton's formula for gravitational acceleration differs from the correct one by less than ten percent? by less than one percent?
4. Compute the weight in pounds of a 100-kilogram astronaut on the surface of a neutron star with mass equal to $1.4M_{\text{Sun}}$ and $M/r_{\text{surface}} = 2/5$.

SOLUTIONS

1. At radius $r/M = 4$ the factor $(1 - 2M/r)^{-1/2}$ in (60) predicts a gravitational acceleration $2^{1/2} = 1.41$ times that predicted by Newton.
2. Even at $r/M = 2.1$ the gravitational acceleration is still the relatively mild multiple of 4.6 times the Newtonian prediction.
3. Setting $(1 - 2M/r)^{-1/2} = 1.1$ yields $r/M = 11.5$. At or outside this radius, Newton's prediction will be in error (too low) by less than ten percent. At or outside the radius $r/M = 100$ Newton's prediction will be too low by one percent.
4. The Newtonian acceleration in conventional units is: (#NeutronNewton)

$$g_{\text{Newton conv}} = \left(\frac{GM_{\text{kg}}}{c^2 r^2_{\text{surface}}}\right) c^2 = \left(\frac{M}{r^2_{\text{surface}}}\right) c^2 \tag{31}$$

$$= \left(\frac{M}{r_{\text{surface}}}\right)^2 \frac{c^2}{M} = \left(\frac{2}{5}\right)^2 \frac{c^2}{1.4 \times M_{\text{Sun}}}$$

Insert values of c^2 and M_{Sun} (in meters) to yield $g_{\text{Newton conv}} \approx 6.9 \times 10^{12}$ meters/second^2. From (60), (#weight)

$$\text{weight} = mg_{\text{shell}} = \left(1 - \frac{4}{5}\right)^{-1/2} mg_{\text{Newton}} \tag{32}$$

$$\approx 16 \times 10^{14} \text{ Newtons}$$

One Newton = 0.225 pounds, so our astronaut weighs approximately 3.5×10^{14} pounds, or 350 million million pounds. It is interesting that even at the surface of this neutron star the general relativity result in (32) is greater than Newton's by the relatively small factor $\sqrt{5} = 2.24$.

Hailstone: hurled inward from infinity.

Drips and raindrops do not exhaust the possibilities for free radial motion. We can also hurl a stone radially inward from a great distance. Call this a **hailstone**, because on Earth a hailstone falls faster than a raindrop. Let the hailstone's initial inward speed at a great distance be v_{far}. Then its energy is that of a stone moving with this speed in flat spacetime, given by the special relativity expression: (#HailEnergy)

$$\frac{E}{m} = \left(1 - v^2_{\text{far}}\right)^{-1/2} \qquad \text{(hailstone)} \tag{40}$$

The hailstone's shell energy at any radius r is obtained from equations (25) and (40): (#HailShellEnergy)

$$\frac{E_{\text{shell}}}{m} = \left(1 - v^2_{\text{far}}\right)^{-1/2} \left(1 - \frac{2M}{r}\right)^{-1/2} \qquad \text{(hailstone)} \tag{41}$$

To find the map velocity dr/dt for the hailstone, equate E/m in (40) to the general expression for energy (10). Solve the resulting equation for $d\tau^2$: (#dtau2Hail)

$$d\tau^2 = \left(1 - v^2_{\text{far}}\right) \left(1 - \frac{2M}{r}\right)^2 dt^2 \qquad \text{(hailstone)} \tag{42}$$

SAMPLE PROBLEM 4. Gravitational Acceleration Near Different Black Holes

How does the predicted initial gravitational acceleration g_{shell} vary with the mass M of a black hole? Examine different cases at the radius $r = 4M$.

1. The smallest mass of a cold dead star that can collapse into a black hole is estimated to be approximately 2.5 times the mass of our Sun. What is the gravitational acceleration at $r = 4M$ near this minimum-mass black hole? This prediction is how many times the gravitational acceleration on Earth's surface?
2. The monster black hole near the center of our galaxy has an estimated mass equal to 3.7 million times the mass of our Sun. Assume (probably wrongly) that the monster is not rotating. What is the gravitational acceleration at $r = 4M$ near this black hole? This prediction is how many times the gravitational acceleration on Earth's surface?
3. How many times the mass of our Sun would a black hole have to be so that the gravitational acceleration on the shell at $r = 4M$ is greater than than on Earth's surface by ten percent?

SOLUTIONS

At $r = 4M$, equation (59) becomes: (#shellgrav3*)

$$g_{\text{shell}} = -\frac{2^{1/2}}{16M} \quad \text{(initial at } r = 4M\text{)} \qquad (33)$$

This is inversely proportional to the mass M of the black hole. At the surface of Earth: (#gravEarth)

$$g_{\text{Earth}} = -\frac{M_{\text{Earth}}}{r^2_{\text{Earth}}} \approx -10^{-16}\ \text{meter}^{-1} \qquad (34)$$

Our Sun's mass is 1.48×10^3 meters.

1. Substituting 2.5 times the mass of our Sun into (33) yields the value $g_{\text{shell}} = -2.4 \times 10^{-5}$ meter^{-1}, which is 2.4×10^{11} times the acceleration on Earth.
2. At $r = 4M$ outside the black hole in our galaxy the gravitational acceleration is $g_{\text{shell}} = -1.6 \times 10^{-11}$ meter^{-1} or "only" 1.6×10^5 times the acceleration on Earth.
3. Set the left side of (33) equal to $1.1 g_{\text{Earth}}$ and use (34) to obtain a mass equal to $5.4 \times 10^{11} M_{\text{Sun}}$. The numerical coefficient is 5 or 6 times the number of stars in our galaxy.

Equate this expression for $d\tau^2$ to that in the Schwarzschild metric (51), divide through by $(dt)^2$ and solve for $(dr/dt)^2$, yielding (#drdtHail)

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right)^2 \left[\frac{2M}{r} + v^2_{\text{far}}\left(1 - \frac{2M}{r}\right)\right] \qquad \text{(hailstone)} \quad (43)$$

Use conversions (13) to give the shell speed: (#ShellSpeedHail)

$$|v_{\text{shell}}| = \left[\frac{2M}{r} + v^2_{\text{far}}\left(1 - \frac{2M}{r}\right)\right]^{1/2} \qquad \text{(hailstone speed)} \quad (44)$$

Drip, rain, hail cover all free-fall radial motions.

Taken together, the three categories drip, rain, and hail encompass all possible radial speeds of a freely plunging stone. Table 1 summarizes the results. You can check that equations for drips (second column) reduce to those for raindrops (third column) when $r_0 \to \infty$. Similarly, equations for hailstones (fourth column) also reduce to those for raindrops when $v_{\text{far}} \to 0$.

?

Why are the expressions for v_{shell} in Table 1 so COMPLICATED? How can a stone carry out all these calculations as it drops freely toward a center of attraction? A stone is brainless, yet in order to follow equations in the table it must be better at quick computation than we are. Do you seriously believe that spacetime—or anything else—is issuing such

TABLE 1 COMPLETE LIST OF RADIAL PLUNGERS [SPREAD ACROSS PAGE]

	Drip	**Rain**	**Hail**
Name of plunger	drip	raindrop	hailstone
Launch method	dropped from rest at r_0	dropped from rest at $r_0 \to \infty$	hurled inward at speed $v_{\rm far}$ from a great distance
$E_{\rm shell}/m$	$\left(1-\frac{2M}{r_0}\right)^{1/2}\left(1-\frac{2M}{r}\right)^{-1/2}$	$\left(1-\frac{2M}{r}\right)^{-1/2}$	$\left(1-v_{\rm far}^2\right)^{-1/2}\left(1-\frac{2M}{r}\right)^{-1/2}$
Shell speed $\lvert v_{\rm shell}\rvert$	$\left(1-\frac{2M}{r_0}\right)^{-1/2}\left(\frac{2M}{r}-\frac{2M}{r_0}\right)^{1/2}$	$\left(\frac{2M}{r}\right)^{1/2}$	$\left[\frac{2M}{r}+v_{\rm far}^2\left(1-\frac{2M}{r}\right)\right]^{1/2}$
Memory jog:	A leaky faucet drips.	Rain falls from a great height.	Hail falls faster than rain.

complicated directions to the poor stone and that the stone is actually FOLLOWING these instructions?

The stone does not follow equations in Table 1. The stone does not care about the Schwarzschild metric or shell time—or even energy as a constant of motion. The stone can be totally brainless because it obeys the simplest command imaginable: “Go straight for the next microsecond!” This command is all the stone hears. Obeying this command is all the stone does.

We, however, are not satisfied with the local description of motion. We ask more global questions: “What is the entire path followed by the stone?” “What is the stone’s speed everywhere along that path?” Motion for the next microsecond does not answer our global questions. Ask a more complicated question, get a more complicated answer! Whose fault is that? The stone’s fault? Nature’s fault? No, it is our fault. If we were satisfied with local description, we could be as serene and unthinking as the stone.

ALL plungers: $dr/dt \to 0$ at the horizon!

Equations (36) and (43) for dr/dt have some surprising consequences. As the plunger approaches the horizon, as $r \to 2M$, the coefficient $(1-2M/r)$ in these equations goes to zero, so the map velocity dr/dt also goes to zero for all three plungers. The Schwarzschild mapmaker, keeping track of the reduced circumference r as a function of map time t, reckons that every plunger slows down as it approaches the event horizon. As it gets closer and closer to the event horizon at $r = 2M$, its map velocity dr/dt goes to zero. When tracked in these coordinates, the plunger itself reaches the event horizon only after infinite map time t.

?

Impossible! The plunger is propelled ever inward by what we would describe in Newtonian language as a fierce gravitational force. How can you possibly ask us to believe that this force results in the raindrop slowing down? Will someone clinging to the shell just outside the event horizon at $r = 2M$ observe the plunger to decrease speed and settle gently—over an infinite time!—to rest at the horizon? The whole idea is simply impossible to believe!

! ¶

What seems insane resolves itself into a more believable result when we follow this questioner's lead and ask who observes locally this zero speed at the event horizon. The answer is, Nobody! Not the shell observer; not a passing free-fall observer. Nobody near the black hole observes *directly* the velocity whose magnitude goes to zero at the horizon. It is a *map velocity*, an entry on a spreadsheet. Blame the coasting to rest at the horizon on our mapmaker accountant, not on us.

$dr/dt \rightarrow 0$ at the horizon just a spreadsheet entry.

Equations (36) and (43) are merely results of calculations, they are "map velocities." The mapmaker tracks changes dr in the r-coordinate and divides each such change by the corresponding computed change dt in map time. Sure enough, the ratio dr/dt approaches zero as the plunger approaches the event horizon at $r = 2M$.

Nobody directly observes map velocity dr/dt. On the other hand, a shell observer can observe and measure the passage of the plunger. Figure 4 compares dr/dt with v_{shell} for the raindrop. No contrast could be greater than these two results, which show how far we have come from Newton's world of universal time and universal flat Euclidean space.

?

I do not care what one or another observer measures or writes in a notebook. I am interested in REALITY! Stop beating around the bush; does the in-falling raindrop REALLY come to rest at the horizon or not?

! ¶

Already in special relativity we learned to concentrate on predicting the result of an experiment. We were forced to acknowledge, for example, that "the time between two events" and "the velocity of a stone" are not invariants; typically they do not have the same values as measured by different inertial observers in relative motion. In this sense "the real time between two events" and "the real velocity of a particle" have no unique meaning. Similarly, here in general relativity "the velocity at the horizon" must refer to the records of some reference frame; the phrase "real velocity" has no unique meaning. According to the mapmaker, the in-falling object comes to rest at the horizon. Next we find that for the shell observer the falling object passes across the horizon with the speed of light (Figure 4). What a contrast!

BOX 4. Newton Predicts the Black Hole?

It's amazing how much of Newton's mechanics works—sort of—on the stage of general relativity. A stone initially at rest far from a center of attraction plunges radially inward. Or a stone on the surface of Earth or a neutron star is fired radially outward, coming similarly to rest at a great distance. In either case, Newtonian mechanics assigns kinetic and gravitational energies to the stone. The gravitational energy is chosen to be zero at a great distance, and initially the stone out there does not move and so has zero kinetic energy. The total energy is therefore zero, as shown in this Newtonian equation: (#eq:14A)

$$E_{\text{conv}} = 0 = \frac{1}{2}mv_{\text{conv}}^2 - \frac{GM_{\text{conv}}m}{r} \quad \text{(Newton)} \quad (45)$$

where subscript "conv" means conventional units. We can use geometric units even while doing Newton's analysis. Divide through by mc^2 and convert mass M to meters and speed v to a fraction of the speed of light. The result is still Newtonian: (#eq:15A)

$$\frac{E}{m} = 0 = \frac{v^2}{2} - \frac{M}{r} \quad \text{(Newton)} \quad (46)$$

Here E and m each can have any units whatsoever, as long as they are the *same* units for both. Equation (46) shows us the plunging (or rising) speed at any radius: (#eq:17A)

$$v = \left(\frac{2M}{r}\right)^{1/2} \quad \text{(Newton)} \quad (47)$$

which is the same as equation (39) for the shell speed of the raindrop. One can predict from (46) the radius at which the speed is unity, the speed of light. And the predicted radius, $r = 2M$, is that of the black hole horizon. For Newton the speed of light is the **escape velocity** from the horizon.

So does Newton correctly describe black holes? No. The similarities are enchanting but the differences are profound. In the first place, Newton assumes a single universal inertial reference frame and universal time, whereas (39) is true only for the shell distance divided by shell time. A quite different expression, (36) with $r_0 \to \infty$, describes map velocity—map distance dr divided by map time dt for raindrops.

Worse: Newton predicts that a stone launched radially outward from the event horizon with a speed less than that of light will rise some radial distance, then slow, stop without escaping, and fall back. In striking contrast, Einstein predicts that nothing, not even light, can be successfully launched outward from the event horizon (exercise in Chapter 5), and that light launched outward *exactly at* the event horizon will hover there forever (Box 5).

Our shell speeds for drip, rain, or hail can be directed radially outward just as well as inward; energy does not care about direction of motion. We can talk about **escape velocity** (Box 4), the minimum outward-directed speed needed to send a stone to infinity. There is no difficulty with this velocity reversal—which one might equally well call "time reversal"—*until the plunger reaches the horizon at* $r = 2M$.

What happens exactly **at** the horizon?

Table 1 tells us that something decisive happens exactly *at* the horizon. As the plunger approaches the horizon at $r = 2M$, the shell speed approaches the speed of light for all three launch methods: drip, rain, and hail. Try to increase the shell speed at the horizon by hurling the stone in from infinity with greater and greater initial speed v_{far}. You fail. Try to make the speed at the horizon less than the speed of light by releasing the stone one millimeter above the horizon. You still fail: the shell speed of the stone rises to the speed of light at the horizon anyway! Even in general relativity the fastest directly-observable speed remains that of light.

Let the plunger emit a light flash radially outward as she crosses the horizon. Then that light will, in principle, hover at the horizon forever. (Such "hovering" is a knife-edge phenomenon discussed in Chapter 4, so the flash—which by definition has some radial extension—will quickly dissipate.)

?

I am really bothered by the idea of a "material" particle traveling across the event horizon as a particle. The shell observer sees it moving at the speed of light, but it takes light to travel at light speed. Does the particle become a flash of light at the horizon?

!

No, but at the event horizon is a light sphere stationary in map and shell coordinates. In the plunger frame, this light will pass at the usual speed $v_{\text{light}} = 1$. Otherwise you feel nothing special as you cross the event horizon. You certainly do not turn into a flash of light! The idea of a shell observer at the horizon makes sense only as a limiting case. The shell observer just outside the horizon measures the in-falling particle to move at less than the speed of light. *At the horizon* no shell is possible, because the "local acceleration of gravity" increases without limit—equation (59). So no shell observer can be stationed *at* the horizon to verify that the in-falling particle moves at light speed. *At* and *inside* the horizon, dependable measurements can be made by free-fall observers, but cannot be reported to outsiders! See Project 3: Inside the Black Hole.

?

Go back to map coordinates. A serious objection still remains: If all objects as they fall into the black hole come to rest at the horizon as reckoned by the mapmaker, then shouldn't the black hole be eternally surrounded by all the junk that has ever fallen into it, including the star that collapsed to form the black hole in the first place? Our Russian colleagues originally called the black hole a "frozen star." I call it a "frozen junk pile"!

!

It's true that, on the spreadsheet of the mapmaker, all radially in-falling objects coast to rest at the horizon. But this is just a spreadsheet entry; nobody observes directly this coming-to-rest. A remote observer in fact does see something that might be considered evidence for this prediction: the gravitational red shift of light from these objects. As each object approaches the horizon, its emitted light is shifted farther and farther into the red as observed far from the black hole. You can calculate how rapidly this downshift occurs as recorded on the clock of a remote observer (exercise in Chapter 5). Very quickly, light from the object becomes invisible to the eye; the object turns black—thus becoming part of the black hole as far as the remote observer is concerned. You might conclude that stationary black junk is still black! But don't be fooled: nobody can view any stationary junk whatsoever at the horizon.

?

But what about a time-reversed drip? Your analysis implies that the drip has to move outward at the speed of light just to rise up even one millimeter above $r = 2M$! And I thought nothing can go as fast as light.

BOX 5. The Event Horizon

Set $d\tau = 0$ in the Schwarzschild metric (51) for radial motion to show that for radially outgoing light at the horizon, (#HorizonLight)

$$\frac{dr}{dt} = 1 - \frac{2M}{r} \to 0 \qquad (\text{as } r \to 2M) \qquad (48)$$

This is the **event horizon**. Light emitted outwards at $r = 2M$ will hover there, held in place by the enormous gravity of the black hole, providing another name for the event horizon: the **radial light sphere**. Massive particles cannot overtake a light beam, so they can never cross outward through $r = 2M$.

The event horizon at $r = 2M$ separates those events which can causally affect the future of distant observers (namely cause events in the future at $r > 2M$) from those that can *never* do so. Barring quantum mechanics, the event horizon never reveals what is hidden behind it.

What is a black hole? We can now improve our definition: *A black hole is a singularity cloaked by an event horizon.*

In Chapter 6, The Expanding Universe, we will find another kind of horizon, called a **particle horizon**. Some astronomical objects are so far from us that the light they have emitted since they were formed has not yet had time to reach us. In principle, more and more such objects swim into our field of view every day, as our cosmic particle horizon sweeps past them. In contrast to the event horizon, the particle horizon yields up its hidden information to us—gradually!

You're right. Any stone reaching $r = 2M$ can never get outside that radius; it *must* fall to the center. Only light itself can hover at $r = 2M$. See Box 5.

7 ■ OVER THE EDGE: ENTERING THE BLACK HOLE

No jerk. No jolt. A hidden doom.

Except for the singularity at $r = 0$, no feature of the black hole excites more curiosity than the event horizon at $r = 2M$. It is the point of no return beyond which no traveler can find the way back—or even send signals—to the outside world. What is it like to fall into a black hole? No one from Earth has yet experienced it. Moreover, future explorers who do so will not be able to return to tell about it or transmit messages to us about their experience—or so we believe! In spite of the impossibility of receiving a final report, there exists a well-developed and increasingly well-verified body of theory that makes clear predictions about our experience as we approach and cross the horizon of a black hole. Here are some of those predictions.

We are not sucked into a black hole.

We are not "sucked into" a black hole. Unless we get quite close to it, a black hole will no more grab us than Sun grabs Earth. If our Sun should suddenly collapse into a black hole without expelling any mass, Earth and the other planets would continue on their present courses undisturbed (even though perpetual night would prevail!). The Schwarzschild solution would continue then to describe Earth's worldline around our Sun, just as it does now. The exercises of Chapter 4 show that for orbits that stay at radii greater than about 300M Newtonian mechanics predicts the motion to a good approximation. We will find that when we drop to a radius less than $6M$, no stable circular orbit is possible (Chapter 4). Even if we find ourselves at a radius between $6M$ and the horizon at $2M$, we can always escape, given sufficient rocket power. Only when we reach or cross the event horizon are we irrevocably "sucked in," our fate sealed.

No jolt as we cross the horizon.

No special event occurs as we fall through the event horizon. Even when we cross into a black hole at the event horizon $r = 2M$, we experience no shudder, jolt, or jar. True, the tidal forces are ever-increasing as we fall inward, and this increase continues smoothly at the horizon. But we are not suddenly torn apart at $r = 2M$. True also, the factor $(1 - 2M/r)$ in the Schwarzschild metric goes to zero at this radius. But the resulting zero in the time term of the metric and the infinity in the radial term turn out to be singularities of map coordinates r and t, not singularities in spacetime geometry. They do not lead to discontinuities in our experience as we pass through this radius. There are other coordinate systems whose metrics are non-singular at the event horizon (Project 3).

No shell frames inside the horizon.

Inside the horizon there are no shell frames. Outside the horizon of the black hole we have erected, in imagination, a set of nested spherical shells. We say "in imagination" because no known material is strong enough to stand up under the "pull of gravity," which increases without limit as one approaches the horizon from outside (Section 6). Locally such a stationary shell can be replaced by a rocket ship with rockets blasting to keep it stationary with respect to the black hole. Inside the horizon, however, nothing can remain at rest. No stationary shell. No stationary rocket ship, however ferocious the blast of its engines. The material composing the original star, no matter how strong, was unable to resist the collapse that formed the black hole. The same irresistible collapse forbids any stationary structure or any motionless object inside the horizon.

Can send packages inward, not outward.

"Outsiders" can send packages to "insiders." Different inertial frames still move with relative speeds inside the black hole. For example, one traveler may drop from rest just outside the horizon. Another unpowered spaceship may have fallen in from rest at a great distance. A third may be hurled inward from outside the horizon. Light and radio waves can carry messages inward as well. We who have fallen inside the horizon can still see the stars, though with changed directions, colors, and intensities (Chapter 5). Packages and communications sent inward across the horizon? Yes. Outward? No! See Box 5.

Inside the horizon there is an exchange of character between the t-coordinate and the r-coordinate. For an r-coordinate less than the map radius $2M$, the factor $(1 - 2M/r)$ in the Schwarzschild metric becomes negative. In consequence, both the radial part and the time part of this metric reverse signs, making the dt^2 term negative and the dr^2 term positive. Space and time themselves do not exchange roles. Coordinates do: The t-coordinate changes in character from a timelike coordinate to a spacelike coordinate. Similarly, the r-coordinate changes in character from a spacelike coordinate to a timelike one.

Timelike r-coordinate inside horizon means inevitable motion toward center.

What does it mean to say that inside the Schwarzschild radius the r-coordinate "changes character from a spacelike coordinate to a timelike one"? It means that our free-fall frame moves to ever-smaller r with all the inevitability that we ordinarily associate with the passage of time. The explorer in his jet-powered spaceship prior to arrival at $r = 2M$ always has the option to turn on his jets and change his motion from decreasing r (infall) to increasing r (escape). Quite the contrary is the situation once he has allowed himself to fall inside $r = 2M$. Then the further decrease of r represents the passage of time. No command the traveler can give to his jet engine will turn back time, that is reverse the headlong decrease in radius. That unseen power of the world that drags everyone forward in time, willy-nilly, from age twenty to forty and from forty to eighty also drags the rocket in from the coordinate $r = 2M$ to the later value of the "time" coordinate $r = 0$. No human act of will, no engine, no rocket, no force can make time stand still. As surely as cells die, as surely as the traveler's watch ticks away the unforgiving minutes, with equal certainty r drops from $2M$ to 0 with never a halt along the way.

Surf a collapsing galaxy group.

Inside the horizon life goes on. Make a daring plunge into an already existing black hole? No. We and our exploration team want to be still more daring, to follow a black hole as it forms. We go to a multiple-galaxy system so crowded that it teeters on the edge of gravitational collapse. Soon after our arrival at the outskirts, it starts the actual collapse, at first slowly, then more and more rapidly. Soon a mighty cataract thunders (silently!) toward the center from all sides, a cataract of objects and radiation, a cataract of momentum-energy. The matter of the galaxies and with it our group of enterprising explorers pass smoothly across the horizon at the Schwarzschild radius $r = 2M$.

From that moment onward we lose all possibility of signaling to the outer world. However, radio messages from that outside world, light from the familiar stars, and packages fired after us at high speed continue to reach us. Moreover, communications among us explorers take place now as they did before we crossed the horizon. We express our findings to each other in the familiar categories of space and time. With our laptop computers we turn out an exciting journal of our measurements and conclusions. (Our motto: Publish *and* perish.)

After crunch there is no "after."

Nothing rivets our attention more than the tide-producing forces that pull heads up and feet down with ever-increasing tension. Before many years have passed, we can predict, this differential pull will have reached the point where we can no longer survive. Moreover, we can foretell still further ahead and with absolute certainty an instant of total crunch. In that crunch are swallowed up not only the stars beneath us, not only we explorers, but time itself. All worldlines terminate at the singularity. For us an instant comes after which there is no "after."

Project 3 discusses life inside the horizon in greater detail.

8 ■ APPENDIX: GRAVITY ON THE SHELL

Unlimited gravitational acceleration on a shell near the horizon.

Is gravity real or fictitious?

When you stand on a shell near a black hole you experience gravity—a pull downward—just as you do on Earth. On the shell this gravity can be great: near the horizon it increases without limit, as we shall see. On the other hand, inside the back cover we say, "In general relativity, gravity is always a fictitious force which can be eliminated by changing to a frame that is in free fall . . .". Is this "fictitious force" real? Every year many people are injured and killed as a result of falls. Any force that can lead to death is definitely real!

Practical experiment to define gravity

We start by acting like engineers, using a thought experiment to define what we mean by local gravitational acceleration on a shell near a black hole—or on Earth. Following this definition we wheel up the machinery of general relativity to find the magnitude of the newly-defined acceleration experienced by a shell observer.

Figure 7 presents the method of measuring quantities used to define initial gravitational acceleration on a shell. The shell is at map radius r_0. At a shell distance Δy_{shell} below the shell is a platform onto which the shell observer drops a stone. The resulting time Δt_{shell} for the drop is measured as follows:

Specific instructions for experiment to define gravity

1. The shell observer records his clock reading at the instant he drops the stone.
2. When the stone strikes the platform, it fires a laser flash upward to the shell clock.

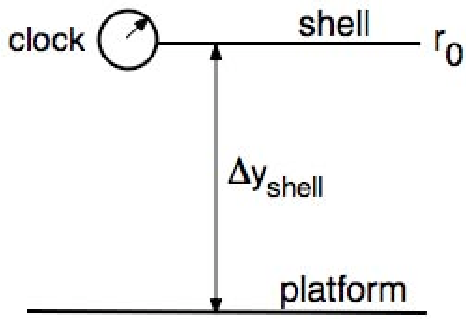

FIGURE 7 Notation for thought experiment to define gravitational acceleration on a shell patch. The shell observer at r_0 releases a stone from rest and times its fall onto a lower platform that he measures to be a distance Δy_{shell} below the shell.

3. The shell observer determines the time lapse to impact, Δt_{shell}, by deducting flash transit time from the time elapsed on his clock when he receives the laser flash.

The shell observer reckons the "flash transit time" in Step 3 by dividing the shell distance Δy_{shell} by the shell speed of light. (In Exercise 00 of Chapter 2, you verified that the shell observer measures light to have its conventional speed: unity.)

Define g_{shell}

The shell observer substitutes Δy_{shell} and Δt_{shell} into the usual expression that defines initial acceleration g_{shell} of the stone: (#yA)

$$\Delta y_{\text{shell}} \equiv \frac{1}{2} g_{\text{shell}} \Delta t_{\text{shell}}^2 \qquad \text{(definition of } g_{\text{shell}} \text{ initial)} \tag{49}$$

Thus far our engineering definition of g_{shell} has little do with general relativity. The fussy procedure used in the thought experiment reflects the care required when general relativity is added to the analysis, which we do now.

Mapmaker demands falling stone have constant energy.

What does the Schwarzschild mapmaker have to say about the acceleration of a dropped stone? She insists that, whatever motion the free stone executes, its energy E/m must remain a constant of motion. So start with the energy of a stone bolted to the shell at r_0, equation (**??**).

Now release the stone from rest. The mapmaker insists that as the stone falls its energy remain constant, so equate the right sides of (**??**) and (10), square the result, and solve for $d\tau^2$: (#equateenergy)

$$d\tau^2 = \left(1 - \frac{2M}{r_0}\right)^{-1} \left(1 - \frac{2M}{r}\right)^2 dt^2 \tag{50}$$

Substitute this expression for $d\tau^2$ into the Schwarzschild metric for radial motion ($d\phi = 0$) (#Schwarzrad)

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \tag{51}$$

Solve the resulting equation for $(dr/dt)^2$: (#drdt)

$$\left(\frac{dr}{dt}\right)^2 = \left(1-\frac{2M}{r_0}\right)^{-1}\left(1-\frac{2M}{r}\right)^2\left(\frac{2M}{r}-\frac{2M}{r_0}\right) \quad \text{(from rest at } r_0\text{)} \tag{52}$$

We want the acceleration of the stone in Schwarzschild map coordinates. First, multiply both sides of (52) by the constant denominator $(1-2M/r_0)$ on the right (the initial drop-radius r_0 does not change during the fall). Then take the time derivative of both sides. Cancel the common factor $2(dr/dt)$ from both sides of the result to obtain: (#dr2dt2)

$$\left(\frac{d^2r}{dt^2}\right) = \left(-\frac{M}{r^2}\right)\left(1-\frac{2M}{r}\right)\left(1-\frac{2M}{r_0}\right)^{-1}\left(\frac{4M}{r_0}+1-\frac{6M}{r}\right) \tag{53}$$

Acceleration in map coordinates

This equation gives the map acceleration at radius r of a stone released from rest at r_0. What we want is the *initial* acceleration at the instant of release from rest. To find the initial acceleration, set $r = r_0$ in equation (53), which then collapses into the relatively simple form: (#bkkpraccel)

$$\frac{d^2r}{dt^2} \equiv -\frac{M}{r_0^2}\left(1-\frac{2M}{r_0}\right) \qquad \text{(initial, from rest at } r_0\text{)} \tag{54}$$

What is the meaning of this acceleration in Schwarzschild map coordinates? It is a spreadsheet entry, an accounting analysis by the mapmaker, not the result of a direct observation by anyone. Observation requires an experiment, which we have already designed, leading to the expression (49). What is the relation between our engineering definition of acceleration and acceleration (54) in Schwarzschild coordinates? To compare the two expressions, expand the Schwarzschild position of the dropped stone around the radial position r_0 using a Taylor series for a short time lapse Δt: (#TaylorA)

$$r = r_0 + \left(\frac{dr}{dt}\right)_{r_0}\Delta t + \frac{1}{2}\left(\frac{d^2r}{dt^2}\right)_{r_0}(\Delta t)^2 + \frac{1}{6}\left(\frac{d^3r}{dt^3}\right)_{r_0}(\Delta t)^3 + ... \tag{55}$$

Because Δt is small, we disregard terms higher than quadratic in Δt. This allows us to approximate uniform gravity and to compare mapmaker accounting entries with observed shell acceleration. When dropped from rest at r_0, the initial speed is zero: $(dr/dt)_{r_0} = 0$. With these considerations, insert (54) into (55) and write: (#bkkpraccelB)

$$r - r_0 = \Delta r \approx \frac{1}{2}\left[-\left(1-\frac{2M}{r_0}\right)\frac{M}{r_0^2}\right](\Delta t)^2 \tag{56}$$

This equation has a form similar to that of our experimental definition (49) of shell gravitational acceleration, except the earlier equation employs shell separation and shell time lapse. Convert these to Schwarzshild quantities using standard transformations: (#Schwarztransf)

$$\Delta y_{\mathrm{shell}} = \left(1 - \frac{2M}{r_0}\right)^{-1/2} \Delta r \quad \text{and} \quad \Delta t^2_{\mathrm{shell}} = \left(1 - \frac{2M}{r_0}\right) (\Delta t)^2 \qquad (57)$$

With these substitutions, and after rearranging terms, equation (49) becomes: (#yAB)

$$\Delta r \equiv \frac{1}{2} \left[\left(1 - \frac{2M}{r_0}\right)^{3/2} g_{\mathrm{shell}}\right] (\Delta t)^2 \qquad (58)$$

As we go to the limit $\Delta t \to 0$, the extra terms in (55) become increasingly negligible, so (56) approaches an equality and we can equate square-bracket expressions in (56) and (58). Replacing r_0 with r yields the equation for initial acceleration on a shell at any r: (#shellgrav)

Initial shell acceleration

$$g_{\mathrm{shell}} = -\left(1 - \frac{2M}{r}\right)^{-1/2} \frac{M}{r^2} \qquad \text{(initial)} \qquad (59)$$

Check limiting cases: At large radius, g_{shell} in (59) approaches the Newtonian expression $-M/r^2$. As we approach the horizon from larger radius, $r \to 2M$, the gravitational acceleration on the shell increases without limit. If you try to stand on a neutron star, you will be crushed by local acceleration g_{shell} (Sample Problem 3).

In (59) the expression $-M/r^2$ is the Newtonian result, so that equation can be written (#shellgravNewt)

$$g_{\mathrm{shell}} = \left(1 - \frac{2M}{r}\right)^{-1/2} g_{\mathrm{Newton}} \qquad \text{(initial)} \qquad (60)$$

The units of g_{shell} on the left side of this equation are the same as the units you choose for g_{Newton} on the right. Sample Problems 3 and 4 explore shell accelerations under different conditions. It is surprising how accurate Newton's expression is even quite close to the horizon of a black hole—an intellectual victory that we could hardly have anticipated.

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PROBLEMS

1. Plunging from Rest at Infinity

Black Hole Alpha has a mass $M = 5$ kilometers and a horizon at $r = 2M = 10$ kilometers. A stone starting from rest far away falls radially into Black Hole Alpha.

A. At what velocity does a shell observer at $r = 35$ kilometers measure the stone to be going as the stone passes him? (Answer to nearest digit is -0.5. Supply answer to three significant figures.)

What is the map velocity dr/dt of the stone as it passes $r = 35$ kilometers? (Answer to one significant figure is -0.4. Supply three-digit accuracy.)

B. At what velocity does a shell observer at $r = 25$ kilometers measure the stone to be going as it passes him? (Answer to one significant figure is -0.6. Supply answer to three significant figures.)

What is the map velocity dr/dt of the stone as it passes $r = 25$ kilometers? (Answer to one significant figure is -0.4. Supply answer to three significant figures.)

C. Qualitatively, what do the formulas in the text lead you to *expect* about the relative shell speeds (greater or smaller) at the two radii? the relative values of the shell and map speeds (greater or smaller) at each radius?

D. In the limit as $r \to 2M$, what is the shell speed of the stone? What is the map speed of the stone?

2. Maximum map Speed

A stone is released from rest far from a black hole of mass M. The stone drops radially inward. Mapmaker records show that the stone's inward speed initially increases but declines toward zero as the stone approaches the horizon. The map speed must therefore reach a maximum at some intermediate radius r. Find this radius for maximum map speed. Find the *value* of the map speed at that radius. Check your answers visually in Figure 7. *Optional, probably hard:* Find the radius of maximum map speed for the more general case of a stone *hurled* into the black hole (Sample Problem 3). Verify that your result reduces to the dropped-from-rest expression when the initial speed is zero.

3. Hitting a Neutron Star

A typical neutron star has a mass equal to approximately 1.4 times the mass of Sun (magnitude well-known observationally) and a radius of roughly 10 kilometers (magnitude not well-known). A stone falls from rest at a great distance onto the surface of a nonrotating neutron star with these values of radius and mass.

A. If this neutron star were a black hole, what would be the r-value of its horizon? What fraction is this of the radius of the neutron star?

B. With what speed does the stone hit the surface of the neutron star as measured by someone standing (!) on the surface?

C. With what speed does the stone hit the surface in map coordinates?

D. With what kinetic energy per unit mass does the stone hit the surface according to the surface observer?

E. What is the energy per unit mass of the stone as it hits the surface according to the mapmaker? (Gotcha!)

F. With what speed and kinetic energy per unit mass does the stone hit the surface according to Newton? Compare with your results of parts B through D.

4. Timetable to the Center

An astronaut drops from rest off a shell of radius r_0. How long a time elapses, as measured on her wristwatch, between letting go and arriving at the center of the black hole? If she jumps off the shell just outside the horizon, what is her horizon-to-crunch time (the maximum possible free-fall horizon-to-crunch time).

Several hints: The first goal is to find $dr/d\tau$, the rate of change of r-coordinate with wristwatch time τ, in terms of r and r_0. Then form an integral whose variable of integration is r/r_0. The limits of integration are from $r/r_0 = 1$ (the release point) to $r/r_0 = 0$ (the center of the black hole). The integral is

$$\tau = -\frac{r_0^{3/2}}{(2M)^{1/2}} \int_1^0 \frac{(r/r_0)^{1/2}\, d\,(r/r_0)}{(1 - r/r_0)^{1/2}} \tag{61}$$

Solve this integral using tricks, nothing but tricks: Simplify by making the substitution $r/r_0 = \cos^2\psi$ (The "angle" ψ is not measured anywhere; it is simply a variable of integration.) Then $(1 - r/r_0)^{1/2} = \sin\psi$ and $d(r/r_0) = -2\cos\psi\sin\psi\, d\psi$ The limits of integration are from $\psi = 0$ to $\psi = \pi/2$ With these substitutions, the integral for proper time becomes

$$\begin{aligned} \tau &= 2\frac{r_0^{3/2}}{(2M)^{1/2}} \int_0^{\pi/2} \cos^2\psi d\psi \\ &= 2\frac{r_0^{3/2}}{(2M)^{1/2}} \left[\frac{\psi}{2} + \frac{\sin 2\psi}{4}\right]\Bigg|_0^{\pi/2} \end{aligned} \tag{62}$$

The answer follows immediately. Its units are meters of light-travel time. Now convert this result to seconds and examine the special case of release from just outside the horizon.

ADD PROBLEM: Show that the three kinds of radial launch of a stone given in equations (16) through (18) yield all possible shell speeds $|v_{\text{shell}}|$ from zero to the value one.

ADD PROBLEM: Show that the three kinds of radial launch of a stone given in equations (16) through (18) yield the *same* shell speed, namely $|v_{\text{shell}}| = 1$ as a limiting case when the stone crosses the horizon. You have shown that at the horizon: (a) you cannot make the observed speed of a stone *greater* than that of light, no matter how fast you hurl it from a great distance and (b) you cannot make the speed of the stone*less* than that of light, no matter how close to the horizon you release it.