Lie Algebras - A Walkthrough

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Part 1: Basics.

1 Introduction.

This article is meant to provide a quick reference guide to Lie algebras: the terminology, important theorems, and a brief overview of the subject. Physicists usually call the elements of Lie algebras *generators*, as for them they are merely differentials of trajectories, tangent vector fields generated by some operators. Thus the distinction between Lie groups and Lie algebras sometimes gets lost. It is the distinction between manifolds and their tangent spaces. If terms as commutator, adjoint or representation in general are used, which apply to both, it is often unclear which of them is meant. The underlying connection is Noether's theorem, which establishes a correspondence between physical invariants and symmetric groups, Lie groups. The approximation of curved objects - the Lie group elements - by first order approximations - the Lie algebra elements - is a standard procedure in physics, which might partially explain the neglect. However, the following lays the emphasis on the algebra part from a terminological point of view. The corresponding concept for groups will be named whenever there is an appropriate one. I cannot write another textbook about Lie algebras here, and there is no need to, as there are already many excellent ones! Instead we will focus on the definitions and theorems, driven by the importance Lie algebras have to physics.

Lie algebras are algebras are vector spaces. They have an internal multiplication, the commutators, as well as a scalar multiplication by elements of the underlying field - and right in the middle of some common misconceptions we are.

Definition: A **Lie algebra** $\mathfrak{g}$ is a vector space over a field $\mathbb{F}$ with a $\mathbb{F}-$bilinear multiplication

$$
\begin{aligned}
[.,.] \,:\, &\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \\
&[X,X] = 0 \\
&[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0
\end{aligned}
$$

The second equation is called **Jacobi-identity**. It's practically nothing else as the product or Leibniz rule of differentiation for the Lie product:

$$
\begin{aligned}
\vartheta_X([Y,Z]) &= [\vartheta_X(Y),Z] + [Y,\vartheta_X(Z)] \\
D_X(Y \cdot Z) &= D_X(Y) \cdot Z + Y \cdot D_X(Z)
\end{aligned}
$$

With the first equation we already have the first difficulty:

$$[X,X] = 0 \Longrightarrow [X,Y] + [Y,X] = 0 \Longrightarrow 2 \cdot [X,X] = 0 \not\Longrightarrow [X,X] = 0$$

The last implication is not valid, if $\operatorname{char} \mathbb{F} = 2$, which is why we have to use the stronger condition for **anti-commutativity** in the definition, and why many chapters in Lie algebra books require fields of characteristic 0 or at least not two, such that the term *anti-communitivity* for

$$[X,X] = 0 \Longleftrightarrow [X,Y] = -[Y,X]$$

actually makes sense. The next big restriction for the field is its algebraic closure. Although the theory of Lie algebras doesn't require an algebraic closed scalar field - and many real Lie algebras are important - it is more than convenient as soon as a Lie algebra is a matrix algebra, i.e. a vector space of linear transformations, or when dealing with representations, roots or weights, because all these involve eigenvalues. The existence of all eigenvalues in general, however, requires an algebraic closed field, simply to get all roots of characteristic polynomials.

For this reason and unless stated otherwise, we assume as scalar field the complex numbers. We also only consider finite dimensional Lie algebras.

The product $[X,Y]$ is called **commutator** of X and Y, and at prior has nothing to do with commutation and commuting X,Y. It is simply the Lie multiplication. So why is it called commutator then? This has a couple of reasons

- There are historical reasons. The theory of Lie groups and algebras have been developed at the end of the 19th century, beginning of the 20th. Emmy Noether could and did already use the works of Lie and

Engel. E.g. the thesis of Engel (1883) was titled: 'On The Theory of Touching Transformations'. Nobody at this time had fields of prime characteristic in mind, and the examples they thought of have all been **linear Lie algebras**. Those are subalgebras of the general linear Lie algebra $\mathfrak{gl}(V)$ of all linear transformations on a real or complex vector space V.

- If we have a linear Lie algebra, also called a matrix algebra, then the commutator is indeed defined as $\mathrm{ad}(X)(Y) = [X,Y] = XY - YX$, i.e. by using the given associative product, here of matrices.
- The commutator in groups is given by $[g,h] = g^{-1}h^{-1}gh$ and commuting group elements are those with $[g,h] = 1$ which means $gh = hg$. Thus it is somehow natural to call transformations which obey $[X,Y] = 0$ that is $XY = YX$ also commuting. In a way, the commutator measures the distance of a product to commutativity.
- The theorem of Ado says, that for every finite dimensional, real or complex Lie algebra $\mathfrak{g}$ there is a natural number $n \in \mathbb{N}$ and a Lie subgroup $G \subseteq GL(n, \mathbb{F})$ such that $\mathfrak{g}$ is isomorphic to the Lie algebra of G . This means, that the linear Lie algebras are the only relevant case for finite dimensional, real or complex Lie algebras.

This essay attempts to provide an overview of Lie algebras and how their classification problem is solved. It certainly cannot substitute a textbook on Lie algebras. We will make some general assumptions for the sake of simplicity and because we want to address the mathematical background of what is used, e.g. in quantum field theory:

- A Lie algebra $\mathfrak{g}$ in our context is finite dimensional and as a vector space complex, or real if explicitly stated. So the scalar field $\mathbb{F} \in \{\, \mathbb{R}, \mathbb{C}\,\}$.
- Although not necessary, we can always have the example $[X,Y] = XY - YX$ as commutator in mind, i.e. assume the presence of a second, associative multiplication on the same vector space of linear transformations. However, it is not automatically another, second algebra structure on the vector space, because we do not require that this associative multiplication is closed, i.e. ends up within $\mathfrak{g}$.
  E.g. the matrices $\mathfrak{sl}(n)$ with vanishing trace form a Lie algebra, although their associative product isn't closed:

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \; , \; \left[\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

- A commutator in groups is always defined as $[x, y] = x^{-1}y^{-1}xy$ and we will have $[X, Y] = XY - YX$ in the Lie algebra. So $[X, Y] = 0 \iff XY = YX$, and for a set $S \subseteq \mathfrak{g}$ we write, e.g.

$$[X, S] = \{\, [X, Y] \,:\, Y \in S \,\} \text{ or } [\mathfrak{g}, \mathfrak{g}] = \{\, [X, Y] \,:\, X, Y \in \mathfrak{g} \,\}$$

We will need some basic vocabulary to outline the theory. I put these basic definitions in a table, such that they can always be looked up if necessary.

2 Vocabulary

	Lie algebra $\mathfrak{g}$		(Lie) Group G
Abelian	$[X, Y] = 0$	Abelian	$[g, h] = g^{-1}h^{-1}gh = 1$
ideal $\mathfrak{I}$	$[\mathfrak{g}, \mathfrak{I}] \subseteq \mathfrak{I}$	normal subgroup N	$[G, N] \subseteq N$ $g^{-1}Ng \subseteq N\ ,\ g^{-1}ng \in N$
center	$\mathfrak{Z}(\mathfrak{g}) = \{X\ :\ [X, \mathfrak{g}] = 0\}$	center	$Z(G) = \{g\ :\ [g, G] = 1\}$
centralizer of $S \subseteq \mathfrak{g}$	$C_{\mathfrak{g}}(S) = \{X\ :\ [X, S] = 0\}$	centralizer of $S \subseteq G$	$C_G(S) = \{g\ :\ [g, S] = 1\}$
normalizer of $S \subseteq \mathfrak{g}$	$N_{\mathfrak{g}}(S) = \{X\ :\ [X, S] \subseteq S\}$	normalizer of $S \subseteq G$	$N_G(S) = \{g\ :\ [g, S] \subseteq S\}$
adjoint repres.	$\mathrm{ad}\ :\ \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ $\mathrm{ad}(X)(Y) = [X, Y]$	adjoint repres.	$\mathrm{Ad}\ :\ G \longrightarrow GL(\mathfrak{g})$ $\mathrm{Ad}(g)(Y) = gYg^{-1}$
homo-morphism	$\varphi\ :\ \mathfrak{g} \longrightarrow \mathfrak{g}$ $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$	homo-morphism	$\varphi\ :\ G \longrightarrow G$ $\varphi(g \cdot h) = \varphi(g) \cdot \varphi(h)$
derivation	$\vartheta([X, Y]) =$ $[\vartheta(X), Y] + [X, \vartheta(Y)]$	w/o	(differential of an automorphism)
inner derivation	$\vartheta = \mathrm{ad}(Z)\ :\ X \mapsto [Z, X]$ for a $Z \in \mathfrak{g}$	w/o	(differential of a conjugation)
derived algebra	$\mathfrak{g}^0 = \mathfrak{g}^{(0)} = \mathfrak{g}$ $\mathfrak{g}^1 = \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$	commutator group	$G^0 = G^{(0)} = G$ $G^1 = G^{(1)} = [G, G]$

	Lie algebra $\mathfrak{g}$		(Lie) Group G
descending central series	$\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}]$	lower central series	$G^n = [G, G^{n-1}]$
derived series	$\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$	derived series	$G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$
$\mathfrak{g}$ is nilpotent	$\mathfrak{g}^n = 0$ for some n	G is nilpotent	$G^n = 1$ for some n
$\mathfrak{g}$ is solvable	$\mathfrak{g}^{(n)} = 0$ for some n	G is solvable	$G^{(n)} = 1$ for some n
nilradical $\mathfrak{N}(\mathfrak{g})$	maximal nilpotent ideal	nilradical $N(G)$	maximal nilpotent normal subgroup
radical $\mathfrak{R}(\mathfrak{g})$	maximal solvable ideal	radical $R(G)$	maximal solvable normal subgroup
$\mathfrak{g}$ is simple	$\mathfrak{g}$ has no proper ideals	G is simple	G has no proper normal subgroups
$\mathfrak{g}$ is semisimple	$\mathfrak{R}(\mathfrak{g}) = 0$	G is semisimple	*)
$\mathfrak{g}$ is reductive	$\mathfrak{R}(\mathfrak{g}) = \mathfrak{Z}(\mathfrak{g})$	G is reductive	**)
$\mathfrak{h} \leq \mathfrak{g}$ is toral	$\operatorname{ad} H$ ($H \in \mathfrak{h}$) are simultaneously diagonalizable	$H \leq G$ torus	linear algebraic group consisting of diagonal matrices

*) A connected linear algebraic group G over an algebraically closed field is called semisimple if every smooth connected solvable normal subgroup of G is trivial.

**) A connected linear algebraic group G over an algebraically closed field is called reductive if every smooth connected unipotent ($\sim$ upper triangular matrices with $1'$s on the diagonal), normal subgroup of G is trivial.

Cartan subalgebra (CSA) $\mathfrak{h} \leq \mathfrak{g}$	$\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$ and $\mathfrak{h}$ is nilpotent
Borel subalgebra $\mathfrak{B} \leq \mathfrak{g}$	maximal solvable subalgebra
Engel subalgebra $\mathfrak{E}_{\mathfrak{g}}(X)$	$\{\, Y \in \mathfrak{g} \,:\, (\mathrm{ad}(X))^n(Y) = 0$ for some $n \in \mathbb{N} \,\}$
structure constants a_{ij}^k	$[X_i, X_j] = \sum_{k=1}^n a_{ij}^k X_k$, (X_k) basis
symmetric bilinear form β	$\beta \,:\, \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{F} \in \{\, \mathbb{R}, \mathbb{C} \,\}$ $\beta(X,Y) = \beta(Y,X)$ $\beta(aX + bY, Z) = a\beta(X,Z) + b\beta(Y,Z)$
radical of β	$\{\, X \in \mathfrak{g} \,:\, \beta(X,Y) = 0$ for all $Y \in \mathfrak{g} \,\}$
Killing-form K	$K(X,Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$

3 Classical (simple) Lie Algebras.

The following subalgebras of $\mathfrak{gl}(V)$ of linear transformations on a finite, $n-$dimensional vector space V are called the classical Lie algebras $\mathfrak{g}$. They are all simple, and plus five exceptional Lie algebras $(E_6, E_7, E_8, F_4, G_2)$ all simple ones there are. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra of dimension l.

Let us further define e_{ij} as the matrix whose entry in the i-th row and j-th column is 1 and 0 elsewhere.

3.1 Special Linear Lie Algebra.

Type: A_l , $\mathfrak{sl}(n,\mathbb{F})$, $\dim \mathfrak{sl}(n,\mathbb{F}) = n^2 - 1 = l^2 + 2l$, $\dim V = n = l + 1$

$\mathfrak{sl}(n,\mathbb{F}) = \{\, X \in \mathbb{M}(n,\mathbb{F}) \,:\, \mathrm{tr}(X) = 0 \,\}$ are all linear transformations on V, i.e. $n \times n$ matrices with vanishing trace. It is thus of dimension $n^2 - 1 = l^2 + 2l$.

Basis: $e_{i\,j}\,(i \neq j)$, $h_i = e_{i\,i} - e_{i+1\,,\,i+1}\,(1 \leq i \leq l)$

The 'special unitary' Lie algebras $\mathfrak{su}(n,\mathbb{C})$ of skew-Hermtian complex matrices with trace 0 are of this type. There is a complex basis transformation of the real vector spaces

$$\mathfrak{su}(n,\mathbb{C}) \cong_{\mathbb{C}} \mathfrak{sl}(n,\mathbb{R})$$

This means they are the same real Lie algebra. The basis transformation, however, needs to be complex, as the skew-Hermitian matrices contain complex numbers. The misleading name is inherited from the group.

Group		Lie Algebra	
special unitary	$SU(n,\mathbb{C})$	'special unitary'	$\mathfrak{su}(n,\mathbb{C})$
special	$\det U = 1$	special	$\operatorname{tr} X = 0$
unitary	$U \cdot U^\dagger = 1$	skew-Hermitian	$X + X^\dagger = 0$

We often find $\mathfrak{sl}(n,\mathbb{R})$ as examples in textbooks about Lie algebras, e.g. to demonstrate their representations. These examples are automatically examples for $\mathfrak{su}(n,\mathbb{C})$, too, modulo some minor adjustments due to the different bases. E.g. the Pauli matrices, which are not skew-Hermitian and thus not elements of $\mathfrak{su}(2,\mathbb{C})$ are all elements of $\mathfrak{sl}(2,\mathbb{C})$. However, their multiples with i are skew-Hermitian. We get them from our basis as

$$\sigma_1 = e_{12} + e_{21} \ , \ \sigma_2 = -ie_{12} + ie_{21} \ , \ \sigma_3 = e_{11} - e_{22}$$

3.2 Orthogonal Lie Algebra On Odd Dimensional Spaces.

Type: $B_l \ , \ \mathfrak{o}(n,\mathbb{F}) \ , \ \dim \mathfrak{o}(n,\mathbb{F}) = \dfrac{n^2-n}{2} = 2l^2 + l \ , \ \dim V = n = 2l+1$

Let β be the nondegenerate, symmetric bilinear form on V whose matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{bmatrix}$. Then the orthogonal algebra is

$$\mathfrak{o}(2l+1,\mathbb{F}) = \mathfrak{o}(V) = \{ \, X \in \operatorname{Hom}(V) \ : \ \beta(X(v),w) + \beta(v,X(w)) = 0 \, \}$$

Basis: (according to the choice of β)

$$\begin{aligned} e_{i\,i} - e_{l+i\,,l+i} &\quad (2 \le i \le l+1) \\ e_{1\,,l+i+1} - e_{i+1\,,1} &\quad (1 \le i \le l) \\ e_{1\,,i+1} - e_{l+i+1\,,1} &\quad (1 \le i \le l) \\ e_{i+1\,,j+1} - e_{l+j+1\,,l+i+1} &\quad (1 \le i \ne j \le l) \\ e_{i+1\,,l+j+1} - e_{j+1\,,l+i+1} &\quad (1 \le i < j \le l) \\ e_{l+i+1\,,j+1} - e_{l+j+1\,,i+1} &\quad (1 \le j < i \le l) \end{aligned}$$

3.3 Symplectic Lie Algebra.

Type: C_l , $\mathfrak{sp}(n,\mathbb{F})$, $\dim \mathfrak{sp}(n,\mathbb{F}) = \dfrac{n^2+n}{2} = 2l^2 + l$, $\dim V = n = 2l$

Let β be the nondegenerate, skew-symmetric bilinear form on V whose matrix is $\begin{bmatrix} 0 & I_l \\ -I_l & 0 \end{bmatrix}$. Then the symplectic algebra is

$$\mathfrak{sp}(2l,\mathbb{F}) = \mathfrak{sp}(V) = \{\, X \in \mathrm{Hom}(V) \,:\, \beta(X(v),w) + \beta(v,X(w)) = 0 \,\}$$

Basis: (according to the choice of β)

$$\begin{aligned} e_{i\,i} - e_{l+i\,,l+i} &\quad (1 \le i \le l) \\ e_{i\,,j} - e_{l+j\,,l+i} &\quad (1 \le i \ne j \le l) \\ e_{i\,,l+i} &\quad (1 \le i \le l) \\ e_{i\,,l+j} + e_{j\,,l+i} &\quad (1 \le i < j \le l) \\ e_{l+i\,,i} &\quad (1 \le i \le l) \\ e_{l+i\,,j} + e_{l+j\,,i} &\quad (1 \le i < j \le l) \end{aligned}$$

3.4 Orthogonal Lie Algebra On Even Dimensional Spaces.

Type: D_l , $\mathfrak{o}(n,\mathbb{F})$, $\dim \mathfrak{o}(n,\mathbb{F}) = \dfrac{n^2-n}{2} = 2l^2 - l$, $\dim V = n = 2l$

Let β be the nondegenerate, symmetric bilinear form on V whose matrix is $\begin{bmatrix} 0 & I_l \\ I_l & 0 \end{bmatrix}$. Then the orthogonal algebra is

$$\mathfrak{o}(2l,\mathbb{F}) = \mathfrak{o}(V) = \{\, X \in \mathrm{Hom}(V) \,:\, \beta(X(v),w) + \beta(v,X(w)) = 0 \,\}$$

Basis: (according to the choice of β)

$$\begin{array}{ll} e_{i\,i} - e_{l+i\,,l+i} & (1 \leq i \leq l) \\ e_{i\,,j} \ - e_{l+j\,,l+i} & (1 \leq i \neq j \leq l) \\ e_{i\,,l+j} - e_{j\,,l+i} & (1 \leq i < j \leq l) \\ e_{l+i\,,j} - e_{l+j\,,i} & (1 \leq j < i \leq l) \end{array}$$

4 Exceptional Lie Algebras.

The actual construction of the exceptional Lie algebras uses concepts like Jordan algebras, octonions and their derivation algebras which will lead too far, so let us summarize them as a list:

Lie algebra $\mathfrak{g}$	E_6	E_7	E_8	F_4	G_2
dim CSA	6	7	8	4	2
$\dim \mathfrak{g}$	78	133	248	52	14

Many of these simple Lie algebras contain other simple Lie algebras as subalgebras, e.g.

$$A_1 \subseteq A_2 \subseteq G_2 \subseteq D_4 \subseteq F_4 \subseteq E_6 \subseteq E_7 \subseteq E_8$$

or see the info graphic on Wikipedia for E_8 .

Whenever we speak of semisimple Lie algebras, then we mean a direct sum of these simple ones (therefore the name 'semisimple'): orthogonal, unitary, symplectic, exceptional; in physics often just one of the simple classical ones.

Theorem: A Lie algebra $\mathfrak{g}$ is semisimple if and only if

$$\mathfrak{g} = \bigoplus_{i=1}^{m} \mathfrak{g}_i \tag{1}$$

is a direct sum of simple ideals $\mathfrak{g}_i \trianglelefteq \mathfrak{g}$.

Part 2: Structures.

5 Decompositions.

Lie algebra theory is to a large extend the classification of the semisimple Lie algebras which are direct sums of the simple algebras listed in the previous paragraph, i.e. to show that those are all simple Lie algebras there are. Their counterpart are solvable Lie algebras, e.g. the **Heisenberg algebra** $\mathfrak{H} = \langle X, Y, Z \ : \ [X,Y] = Z \rangle$. They have less structure each and are less structured as a whole as well. In physics they don't play such a prominent role as simple Lie algebras do, although the reader might have recognized, that e.g. the **Poincaré algebra** - the tangent space of the Poincaré group at its identity matrix - wasn't among the simple ones. It isn't among the solvable Lie algebras either like $\mathfrak{H}$ is, so what is it then? It is the tangent space of the Lorentz group plus translations: something orthogonal plus something Abelian (solvable).

Theorem: The radical $\mathfrak{R}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a solvable ideal, $\mathfrak{g}/\mathfrak{R}(\mathfrak{g}) \cong \mathfrak{g}_s \leq \mathfrak{g}$ a semisimple subalgebra and $\mathfrak{g}$ the semidirect product

$$\mathfrak{g} = \mathfrak{R}(\mathfrak{g}) \rtimes \mathfrak{g}_s \cong \mathfrak{R}(\mathfrak{g}) \rtimes \mathfrak{g}/\mathfrak{R}(\mathfrak{g}) \tag{2}$$

This decomposition is one of the reasons why semisimple and solvable Lie algebras are of interest. The classification of the former is done, the one on the solvable part unfortunately is not. This is mainly due to the different complexity of their multiplicative structures, resp. the lack of it, or the different complexity of their representations if you like.

The starting point of any classification is usually the question:

What does it consist of and what is it composed of?

We already know that we may consider the elements of a Lie algebra as linear transformations. This is not really astonishing, as we always have $\operatorname{ad}(\mathfrak{g}) \subseteq \mathfrak{gl}(g)$ which **are** linear transformations, inner derivations to be exact.

$$\ker \operatorname{ad} = \mathfrak{Z}(g)$$

is an ideal, which means $\mathfrak{Z}(g) = 0$ for semisimple Lie algebras, we even have a faithful (injective) representation as linear transformations for (semi-)simple Lie algebras for free. It also implies, that there is no single Lie algebra element in a semisimple Lie algebra, which commutes with all other elements! Nevertheless, commutation is a convenient property, e.g. simultaneously diagonalizable linear transformations commute.

On the level of linear transformations, the terms diagonalizable, semisimple and toral mean practically the same - at least if all eigenvalues are available, i.e. over algebraically closed fields like $\mathbb{C}$.

property	applies to	means
semisimple	linear transformations	all roots of its minimal polynomial are distinct
diagonalizable	matrices	there is a basis of eigenvectors
toral	subalgebra of linear transformations	all elements are semisimple

The classification of semisimple Lie algebras is based on four fundamental insights. We already mentioned that semisimple Lie algebras are a direct sum of simple Lie algebras and vice versa (1). This result isn't the first one in its natural order. In fact one starts with the second from the following list, but this isn't important in our context:

1. $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$ ($\mathfrak{g}$ **semisimple**, $\mathfrak{g}_i$ **simple**)
2. **The Jordan normal form applied on inner derivations.**
3. **The Cartan subalgebras are toral.**
4. **The Killing-form defines angels.**

Of course there are a lot of technical details to get there as well as to combine these results to a theory of semisimple Lie algebras, especially some geometrical considerations now that we have angels. However, this basically is it.

The **decomposition into simple ideals** is extremely helpful, as all inner derivations ($\operatorname{ad} X$) have a block form, and for the Cartan subalgebras we get a corresponding decomposition $\mathfrak{h} = \bigoplus_{i=1}^{n} \mathfrak{h}_i$ into the separate Cartan subalgebras, which allows us to concentrate on simple Lie algebras only.

The **Jordan normal form** is the starting point. As mentioned, this is quite natural as the inner derivations $\operatorname{ad} X$ provide a faithful representation for simple Lie algebras which have no proper ideals, and especially no center.

The Jordan normal form is an additive decomposition of linear transformations in a semisimple (diagonal) part with its eigenvalues, and a nilpotent (upper triangular) part. The algebraic multiplicity k of an eigenvalue is its multiplicity in the characteristic polynomial and the dimension of the generalized eigenspace

$$G_\lambda(X) = \ker(\operatorname{ad} X - \lambda \cdot \operatorname{id}_{\mathfrak{g}})^k = \{\, Y \in \mathfrak{g} \,|\, (\operatorname{ad}(X) - \lambda \cdot \operatorname{id}_{\mathfrak{g}})^k(Y) = 0 \,\}$$

It is important to distinguish the characteristic and the minimal polynomial, as well as the geometric multiplicity of an eigenvalue, which is the dimension of the eigenspace

$$E_\lambda(X) = \ker(\operatorname{ad} X - \lambda \cdot \operatorname{id}_{\mathfrak{g}}) = \{\, Y \in \mathfrak{g} \,|\, (\operatorname{ad}(X) - \lambda \cdot \operatorname{id}_{\mathfrak{g}})(Y) = 0 \,\} \subseteq G_\lambda(X)$$

The geometric multiplicity determines the number of Jordan blocks of the Jordan normal form, the algebraic multiplicity determines the degree of nilpotency of the nilpotent part of a Jordan block, i.e. the number of ones in the upper triangular part of the Jordan normal form.

Theorem (Jordan-Chevalley Decomposition): Let V be a finite dimensional vector space over a field $\mathbb{F}$ and $\varphi : V \longrightarrow V$ an endomorphism. Then there exist unique endomorphisms φ_s , φ_n such that

$$\begin{gathered} \varphi = \varphi_s + \varphi_n \ , \ \varphi_s \text{ is semisimple } \ , \ \varphi_n \text{ is nilpotent } \ , \ [\varphi_s, \varphi_n] = 0 \\ \varphi_s = p(\varphi) \ , \ \varphi_n = q(\varphi) \text{ for some } p(x), q(x) \in \mathbb{F}[x] \text{ with } x \,|\, p(x), q(x) \end{gathered}$$

In particular, φ_s and φ_n commute with any endomorphism commuting with φ. The decomposition $\varphi = \varphi_s + \varphi_n$ is called the additive Jordan-Chevalley decomposition of φ and φ_s, φ_n are called respectively the semisimple and nilpotent part of φ. Moreover,

$$\operatorname{ad} \varphi = \operatorname{ad} \varphi_s + \operatorname{ad} \varphi_n$$

is the Jordan-Chevalley decomposition of $\operatorname{ad} \varphi$.

The semisimple parts play the key role in the classification of semisimple Lie algebras as well as in their representations. Since they are diagonalizable, i.e. there is a basis of eigenvalues, they also play the key role in physics. Another example of the importance of diagonalizable parts is the following theorem.

Theorem (Malcev Decomposition): A solvable, complex Lie algebra $\mathfrak{g}$ can be written as semidirect product

$$\mathfrak{g} = \mathfrak{R}(\mathfrak{g}) = \mathfrak{T} \ltimes \mathfrak{N}(\mathfrak{g}) \tag{3}$$

of a toral subalgebra $\mathfrak{T}$ and its nilradical $\mathfrak{N}(\mathfrak{g})$.

Summary: Let $\mathfrak{g}$ be any finite dimensional, complex Lie algebra, then

$$\underbrace{\underbrace{\mathfrak{gl}(V) \supseteq \mathfrak{g}}_{\text{linear Lie algebra}}}_{\text{Theorem of Ado}} = \underbrace{\left(\underbrace{\mathfrak{N}(\mathfrak{g})}_{\text{nilpotent}} \rtimes^{(3)} \underbrace{\mathfrak{T}}_{\text{toral}} \right)}_{\text{Theorem of Malcev}} \rtimes^{(2)} \underbrace{\left(\bigoplus_{i=1}^{m} \underbrace{\mathfrak{g}_{\mathfrak{i}}}_{\text{simple}} \right)^{(1)}}_{\text{semisimple}}$$

This means, that we now have to decompose the simple Lie algebras, i.e. those with no proper ideals. Again the toral parts are the key for the next decomposition.

Let us assume $\mathfrak{g}$ is a finite dimensional, complex, simple Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a **Cartan subalgera (CSA)**, i.e. a nilpotent and self-normalizing subalgebra. This apparently weird definition of a Cartan subalgebra turns out to be sufficient to derive the following nice properties.

Theorem (CSA):

- Cartan subalgebras are precisely the maximal toral subalgebras.
- Toral subalgebras are Abelian.
- A Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is self-centralizing:
  $C_{\mathfrak{g}}(\mathfrak{h}) = \{ X \in \mathfrak{g} \,|\, [X, \mathfrak{h}] = 0 \} = E_0(\mathfrak{h}) = \mathfrak{h}$
- $\mathrm{ad}(\mathfrak{h})$ is simultaneously diagonalizable.
- All Cartan subalgebras are conjugate under inner automorphisms of $\mathfrak{g}$, the group generated by all $\exp(\mathrm{ad}\, X)$ with $X \in \mathfrak{g}$ ad-nilpotent.

So why isn't $\mathfrak{h}$ defined as a toral subalgebra in the first place? One reason is, that we haven't shown the existence of Cartan subalgebras, and this can easier be done with the given definition. Anyway, we get the useful and central

Theorem (Cartan decomposition or Root Space Decomposition): Let $\mathfrak{g}$ be a (semi)simple Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{4}$$

where $\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \,|\, [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h} \}$ and $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$ are the eigenspaces of all (simultaneously diagonalizable) linear transformations $\mathrm{ad}\, H$, $\alpha \in \mathfrak{h}^*$.

Those linear forms $\alpha \neq 0$ for which $\mathfrak{g}_\alpha \neq \{0\}$ are called **roots** and Φ the **root space** of $\mathfrak{h}$. All $\mathfrak{g}_\alpha\,(\alpha \neq 0)$ are one dimensional, so let E_α be basis vectors. In particular, we have

$$[H, E_\alpha] = \alpha(H) \cdot E_\alpha \text{ for all } H \in \mathfrak{h}\ ,\ \alpha \in \Phi \subseteq \mathfrak{h}^* \tag{5}$$

6 Geometry.

What happens next is not less than a little miracle! We will see that the root space of a simple Lie algebra has some unexpected properties, which in the end enabled their classification. Something which is for solvable and therewith arbitrary Lie algebras far from being achieved. Remember, that this includes examples like the Heisenberg and Poincaré algebra. The best we have for solvable Lie algebras over algebraically closed fields, is that they *stabilize flags*:

Theroem (solvable Lie Algebras): Let $\mathfrak{g}$ be a solvable complex Lie algebra, V an $n-$dimensional vector space, and $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ a Lie algebra homomorphism. Then there is a sequence of subspaces

$$\{0\} \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = V$$

such that $\dim V_k = k$ and $\varphi(\mathfrak{g})(V_k) \subseteq V_k$. This means especially for the left-multiplication $\varphi = \mathrm{ad}$ that we have a sequence of ideals $\mathfrak{I}_k \leq \mathfrak{g}$ with $\dim \mathfrak{I}_k = k$ and

$$\{0\} \lneq \mathfrak{I}_1 \lneq \ldots \lneq \mathfrak{I}_m = \mathfrak{R}(\mathfrak{g}) = \mathfrak{g} \tag{6}$$

We now assume that $\mathfrak{g}$ is always **a simple finite dimensional Lie algebra** and $\mathfrak{h} \subseteq \mathfrak{g}$ **a Cartan subalgebra**. The reader may think of it as one of the classical, simple Lie algebras listed in chapter 3. Our next task will be to investigate these root spaces $\mathfrak{g}_\alpha = \mathrm{span}(E_\alpha)$. E.g. the Jacobi identity and equation (5) yield

$$\begin{aligned}
[H, [E_\alpha, E_\beta]] &= [E_\alpha, [H, E_\beta]] - [E_\beta, [H, E_\alpha]] \\
&= \beta(H) \cdot [E_\alpha, E_\beta] - \alpha(H) \cdot [E_\beta, E_\alpha] \\
&= (\alpha + \beta)(H) \cdot [E_\alpha, E_\beta]
\end{aligned}$$

and thus

$$[E_\alpha, E_\beta] \in \mathbb{F} \cdot E_{\alpha+\beta} \tag{7}$$

and the ladder operators almost shine through.

Next the Killing-form comes into play. It can be shown that **the Killing-form is nondegenerate** if and only if $\mathfrak{g}$ is semisimple, i.e.

$$\{ X \in \mathfrak{g} \,|\, K(X,Y) = \operatorname{tr}(\operatorname{ad} X \circ \operatorname{ad} Y) = 0 \text{ for all } Y \in \mathfrak{g}) \} = \{ 0 \}$$

Furthermore, the Killing-form restricted on $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate, and $K(E_\alpha, E_\beta) = 0$ for all $\alpha, \beta \in \mathfrak{h}^*$ with $\alpha + \beta \neq 0$; in particular $K(\mathfrak{h}, E_\alpha) = 0$.

These are a very strong properties, because it allows us to use certain numbers $K(H, H')$ as scaling factors while large parts are orthogonal with respect to the Killing-form. We first define a correspondence

$$\mathfrak{h}^* \supset \Phi \longleftrightarrow \{ F_\alpha \,:\, \alpha \in \Phi \} \subset \mathfrak{h} \text{ by } \alpha(H) =: K(F_\alpha, H)\ ,\ H \in \mathfrak{h} \tag{8}$$

define on $\mathfrak{h}^*$ the inner product

$$(\alpha, \beta) := K(F_\alpha, F_\beta)$$

and normalize $H_\alpha := \dfrac{2 \cdot F_\alpha}{(\alpha,\alpha)}$ such that equation (5) now reads

$$[H_\alpha, E_\alpha] = \alpha(H_\alpha) \cdot E_\alpha = \frac{2}{(\alpha,\alpha)} \cdot \alpha(F_\alpha) \cdot E_\alpha = 2 \cdot E_\alpha \tag{9}$$

Meanwhile our Lie algebra can be written (as a direct sum of vector spaces)

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha = \operatorname{span}\{H_\alpha \,|\, \alpha \in \Phi\} \oplus \sum_{\alpha \in \Phi} \mathbb{F} \cdot E_\alpha \tag{10}$$

and we already know, that the Cartan subalgebra $\mathfrak{h}$ is Abelian, the one-dimensional eigenspaces $\mathfrak{g}_\alpha$ are simultaneous eigenvectors of the left multiplications $\operatorname{ad}(H)(X) = [H, X]$, two eigenspaces are Killing orthogonal for $\alpha + \beta \neq 0$, and that $\mathfrak{h}$ is spanned by vectors H_α which satisfy equation (9). The *miracle* can be summarized in the following theorem, and especially the third property is essential for what follows.

Theorem (Root System): Let $\alpha, \beta \in \Phi$ be roots such that $\alpha + \beta \neq 0$.

1. $0 \notin \Phi$ is finite and spans $\mathfrak{h}^*$.
2. If $\alpha \in \Phi$ then $-\alpha \in \Phi$ and no other multiple is.
3. If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle := \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$, the **Cartan integers**.

4. If $\alpha, \beta \in \Phi$ then the reflection $\sigma_\alpha(\beta) := \beta - \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \cdot \alpha \in \Phi$.

Remarks:

1. $H_\alpha \triangleq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E_\alpha \triangleq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{-\alpha} \triangleq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ for $\alpha \in \Phi$ build simple subalgebras $\mathfrak{sl}(2)$ of type A_1

$$[H_\alpha, E_\alpha] = 2E_\alpha \ , \ [H_\alpha, E_{-\alpha}] = -2E_{-\alpha} \ , \ [E_\alpha, E_{-\alpha}] = H_\alpha \, .$$

2. $(\alpha, \beta) \in \mathbb{Q}$ is a positive definite, symmetric bilinear form, in other words, an inner product on the real vector space $\mathcal{E}$ spanned by Φ.

3. $\dim \mathcal{E} = l = \dim \mathfrak{h}^* = \dim \mathfrak{h} = \operatorname{rank} \Phi$

At this point we have all ingredients which are necessary: A real Euclidean vector space $\mathcal{E}$ with an inner product (,), reflections σ_α relative to the hyperplane $\mathcal{P}_\alpha = \{ \beta \in \mathcal{E} \,|\, (\beta, \alpha) = 0 \}$, and most of all, integer values for $\langle \beta, \alpha \rangle$, which by the way is only linear in the first argument. However, we can define angles now:

$$\cos\theta = \cos\measuredangle(\alpha,\beta) := \frac{(\alpha,\beta)}{||\alpha|| \cdot ||\beta||} = \frac{(\alpha,\beta)}{\sqrt{(\alpha,\alpha)}\sqrt{(\beta,\beta)}} \tag{11}$$

$$\langle \alpha, \beta \rangle \cdot \langle \beta, \alpha \rangle = 4\cos^2\theta \in \mathbb{N}_0 \tag{12}$$

and we have reduced the classification problem to a geometric problem! What's left is a discussion of equation (12). Note that the major condition to proceed this way was the equivalence of a nondegenerate Killing-form to a direct sum of simple Lie algebras.

We also know already, that for $l = 1$ there is only one possibility $\Phi = \{-\alpha, \alpha\}$: the simple Lie algebra $\mathfrak{sl}(2)$ of type A_1 .

7 Dynkin Diagrams.

The hyperplanes $\mathcal{P}_\alpha \, (\alpha \in \Phi)$ partition $\mathcal{E}$ into finitely many regions; the connected components of $\mathcal{E} - \cup_{\alpha\in\Phi} \mathcal{P}_\alpha$ which are called the (open) **Weyl chambers**. The group generated by the reflections $\sigma_\alpha \, (\alpha \in \Phi)$ is called **Weyl group** $\mathcal{W}$ of Φ .

Let's have a look on $\mathcal{E} = \operatorname{span}\{\Phi\}$ and choose a basis $\Delta = \{ \alpha_1, \ldots, \alpha_l \}$ such that all $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ with integer coefficients k_α.

This can be done such that either all coefficients are positive, in which case we call the root **positive** $\beta \succ 0$, or all coefficients are negative, in which case we call the root **negative** $\beta \prec 0$, and write $\Phi = \Phi^+ \cup \Phi^-$. The roots $\alpha \in \Delta$ are called **simple**, and $\mathrm{ht}(\beta) = \sum_{\alpha\in\Delta} k_\alpha$ the **height** of $\beta \in \Phi$. A root system Φ is called **irreducible** if it cannot be partitioned into two proper orthogonal subsets. Irreducible root systems correspond to simple Lie algebras, i.e. by our assumption that $\mathfrak{g}$ is simple, our root system is irreducible. It can be shown that for an irreducible root system Φ at most two different root lengths can occur and roots of equal length are conjugate under $\mathcal{W}$. In case of two different lengths, we speak of **short roots** and **long roots**, in case of only one root length, it's called long as a convention.

Meanwhile our simple Lie algebra looks like

$$\mathfrak{g} = \underbrace{\mathfrak{h} \oplus \sum_{\alpha\in\Phi^+} \mathbb{F}\cdot E_\alpha}_{\text{solvable Borel subalgebra}} \oplus \underbrace{\sum_{\alpha\in\Phi^-} \mathbb{F}\cdot E_\alpha}_{\text{nilpotent subalgebra}} \tag{13}$$

$$\Phi = \mathrm{span}_{\mathbb{Z}}\,\Delta = \mathrm{span}_{\mathbb{Z}}\{\,\alpha_1,\ldots,\alpha_l\,\} = \underbrace{\Phi^+ \cup \Phi^-}_{\text{partially ordered}} \tag{14}$$

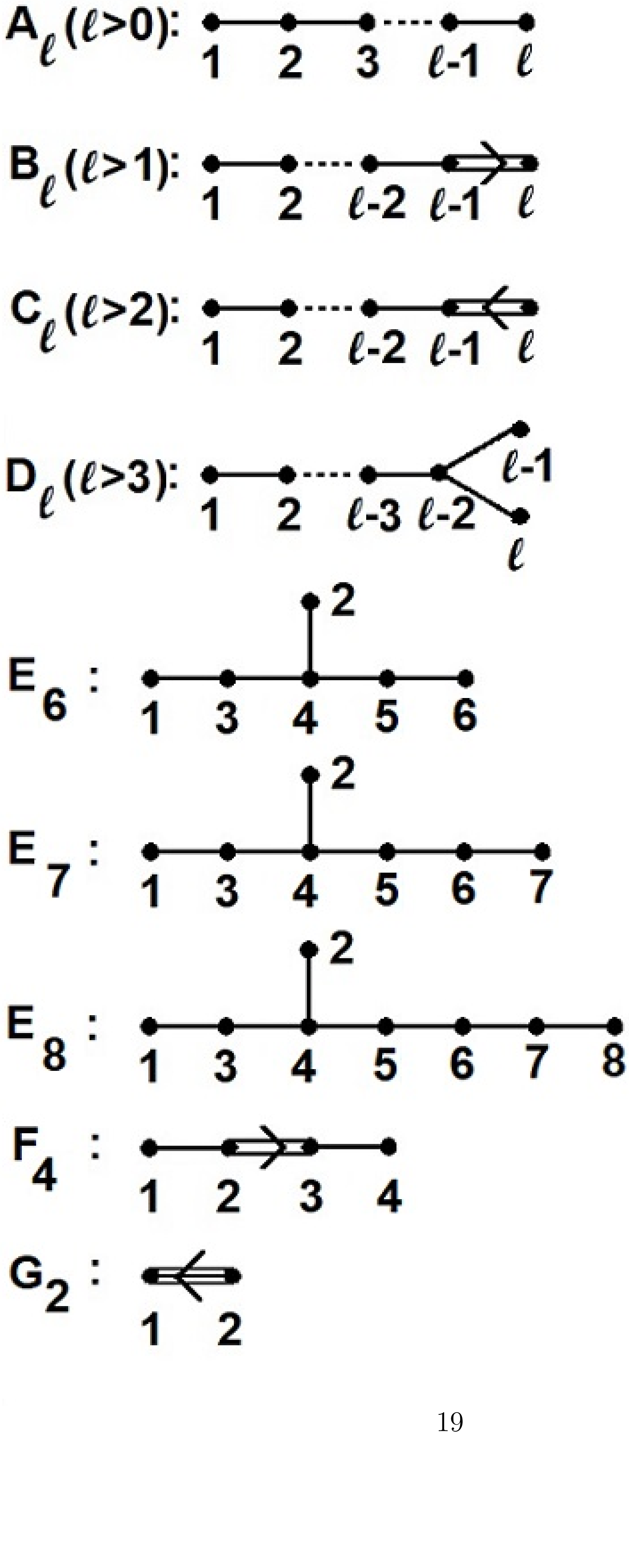
Let's fix an ordering of simple roots $\Delta = \{\,\alpha_1,\ldots,\alpha_l\,\}$. Then the matrix of Cartan integers $(\langle\alpha_i,\alpha_j\rangle)_{i,j}$ is called the **Cartan matrix** of $\mathfrak{g}$.

For distinct positive roots α, β, we have

$$\langle\alpha,\beta\rangle\cdot\langle\beta,\alpha\rangle \in \{\,0,1,2,3\,\}$$

so we can define the **Coxeter graph** of Φ to be a graph with $|\Delta| = l$ vertices and the $i-$th is joined to the $j-$th $(i \neq j)$ by $\langle\alpha_i,\alpha_j\rangle\cdot\langle\alpha_j,\alpha_i\rangle$ many edges. The Coxeter graph completely determines the Weyl group, but it fails to show us in case of two or three edges, which vertex of a pair corresponds to a short simple root and which to a long root. Therefore we add an arrow pointing to the shorter of two roots, whenever there is a double or triple edge. The resulting graph is called **Dynkin diagram** of $\mathfrak{g}$ and allows to recover the Cartan matrix. Irreducible root systems have connected Dynkin diagrams.

Classification Theorem. If $\mathfrak{g}$ is a simple Lie Algebra with an irreducible root system of $\operatorname{rank}\Phi = \dim\mathfrak{h} = l$, then it has one of the following Dynkin diagrams:



Aℓ (ℓ>0):
1 2 3 ℓ-1 ℓ
Bℓ (ℓ>1):
1 2 ℓ-2 ℓ-1 ℓ
Cℓ (ℓ>2):
1 2 ℓ-2 ℓ-1 ℓ
Dℓ (ℓ>3):
1 2 ℓ-3 ℓ-2 ℓ-1 ℓ
E6 :
2
1 3 4 5 6
E7 :
2
1 3 4 5 6 7
E8 :
2
1 3 4 5 6 7 8
F4 :
1 2 3 4
G2 :
1 2

8 Cartan Matrices.

$$A_l : \quad \begin{pmatrix} 2 & -1 & 0 & & & \cdots & & 0 \\ -1 & 2 & -1 & 0 & & \cdots & & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

$$B_l : \quad \begin{pmatrix} 2 & -1 & 0 & & \cdots & & & 0 \\ -1 & 2 & -1 & 0 & \cdots & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

$$C_l : \quad \begin{pmatrix} 2 & -1 & 0 & & \cdots & & & 0 \\ -1 & 2 & -1 & & \cdots & & & 0 \\ 0 & -1 & 2 & -1 & \cdots & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{pmatrix}$$

$$D_l : \quad \begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ -1 & 2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & -1 & 2 & -1 & -1 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 2 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 0 & 2 \end{pmatrix}$$

$E_6:$

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$E_7:$

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$E_8:$

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$F_4:$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$G_2:$

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

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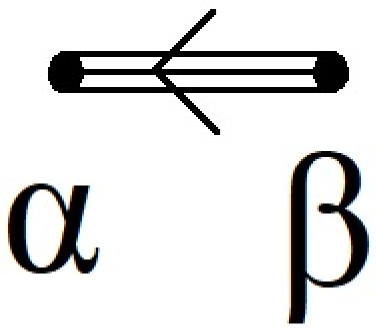
9 Example.

We will show, how Cartan matrices and root systems can be retrieved from the Dynkin diagram on the example of G_2 .



The Dynkin diagram tells us that $\alpha \prec \beta$ and $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 3$. The cosine formula tells us, that the angel they enclose is 30° but this doesn't matter here. Since the only ways to get an integer product of three are $3 \cdot 1 = (-3) \cdot (-1) = 3$ we may assume w.l.o.g. and the sign in the theorem of root systems in mind, that $\langle \alpha, \beta \rangle = -1$ and $\langle \beta, \alpha \rangle = -3$. This produces the Cartan matrix

$$G_2 \, : \, \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Next we calculate by linearity in the first argument

$$\begin{aligned} \alpha - \langle \alpha, \beta \rangle \cdot \beta &= \alpha + \beta \\ \beta - \langle \beta, \alpha \rangle \cdot \alpha &= 3\alpha + \beta \\ (\alpha + \beta) - \langle \alpha + \beta, \alpha \rangle \cdot \alpha &= 2\alpha + \beta \\ (3\alpha + \beta) - \langle 3\alpha + \beta, \beta \rangle \cdot \beta &= 3\alpha + 2\beta \end{aligned}$$

From the decomposition formula in (13) we get with a two dimensional Cartan subalgebra $\mathfrak{h} = \operatorname{span}\{ H_\alpha, H_\beta \}$ the roots

$$\begin{aligned} \Phi^+ &= \{ \alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta \} \\ \Phi^- &= \{ -\alpha, -\beta, -\alpha - \beta, -2\alpha - \beta, -3\alpha - \beta, -3\alpha - 2\beta \} \end{aligned}$$

and

$$G_2 = \operatorname{span}\{ H_\alpha, H_\beta \} \oplus \sum_{\gamma \in \Phi^+ \cup \Phi^-} \mathbb{F} \cdot E_\gamma$$

Part 3: Representations.

10 Sums and Products.

Frobenius began in 1896 to generalize Weber's group characters and soon investigated homomorphisms from finite groups into general linear groups $GL(V)$, supported by earlier considerations from Dedekind. Representation theory was born, and it developed fast in the following decades. The basic object of interest, however, has never been changed: A structure preserving mapping from one class of objects into another which allows matrix *representations.*

Definition: A representation of a (Lie) group G on a vector space V is a (Lie) group homomorphism

$$\begin{gathered}\varphi\,:\,G\longrightarrow GL(V)\\ \varphi(x\cdot y)=\varphi(x)\circ\varphi(y)\end{gathered}$$

Definition: A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism

$$\begin{gathered}\varphi\,:\,\mathfrak{g}\longrightarrow\mathfrak{gl}(V)\\ \varphi([X,Y])=[\varphi(X),\varphi(Y)]=\varphi(X)\circ\varphi(Y)-\varphi(Y)\circ\varphi(X)\end{gathered}\tag{15}$$

This is called a **linear representation** of $\mathfrak{g}$ to be exact. Formally it is the pair (V,φ), but usually only one part is referred to as representation, preferably φ. If V is finite dimensional, then the representation is called **finite dimensional**, if $\ker(\varphi)=\{\,0\,\}$ then the representation is called **faithful** - nothing gets lost. A representation is called **irreducible**, if $\{\,0\,\}$ and V are exactly the only two (under $\varphi(\mathfrak{g})$) invariant subspaces of V, resp. if the $\varphi(X)$ cannot be written as block matrices $\begin{bmatrix}A & B\\ 0 & C\end{bmatrix}$ with the same block structure simultaneously for all $X\in\mathfrak{g}$.

Another notation is: $\mathfrak{g}$ **operates on** V, or V is a $\mathfrak{g}-$**module**

$$X.v:=\varphi(X)(v)\tag{16}$$

They all are simply different wordings of equation (15).

Given two representations (V,φ) and (W,ψ) of $\mathfrak{g}$ we can define other representations by

$$\begin{aligned} \textbf{direct sum } & (V \oplus W, \varphi \oplus \psi) \,:\, \mathfrak{g} \longrightarrow \mathfrak{gl}(V) \oplus \mathfrak{gl}(W) \subseteq \mathfrak{gl}(V \oplus W) \\ \text{as } & (\varphi \oplus \psi)(X)(v+w) = \varphi(X)(v) + \psi(X)(w) \\ & X.(v+w) = X.v + X.w \end{aligned}$$

$$\begin{aligned} \textbf{tensor product } & (V \otimes W, \varphi \otimes \psi) \,:\, \mathfrak{g} \longrightarrow \mathfrak{gl}(V \otimes W) \\ \text{as } & (\varphi \otimes \psi)(X)(v \otimes w) = \varphi(X)(v) \otimes w + v \otimes \psi(X)(w) \\ & X.(v \otimes w) = X.v \otimes w + v \otimes X.w \end{aligned}$$

$$\begin{aligned} \textbf{dual } & (V^*, \varphi^*) \,:\, \mathfrak{g} \longrightarrow \mathfrak{gl}(V^*) \\ \text{as } & \varphi^*(X)(f) = -f(\varphi(X)(v)) \\ & X.f(v) = -f(X.v) \end{aligned}$$

The similarity in the definition of tensor products to the Leibniz rule is no incident: a differential X operating on a certain product $v * w$.

The minus sign in the definition on dual spaces is necessary, since otherwise we would get an anti-homomorphism in (15) due to the rule $f(X.Y.v) = X.f(Y.v) = (X.f)(Y.v) = Y.(X.f(v))$.

A representation is called **completely reducible**, if it can be written as a direct sum of irreducible representations, or equivalently if any invariant subspace $W \subseteq V$ has an invariant complement $W' \subseteq V$ such that $V = W \oplus W'$.

Theorem (Weyl): Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a finite dimensional, semisimple linear Lie algebra, e.g. the simple classical Lie algebras, with finite dimensional vector space V. Then $\mathfrak{g}$ contains the semisimple (diagonal) and nilpotent (upper triangular) parts in $\mathfrak{gl}(V)$ of all its elements.

This theorem has a very important consequence. Let us consider the Jordan decomposition

$$\mathrm{ad}(X) = \mathrm{ad}(X_s) + \mathrm{ad}(X_n)$$

Then $X = X_s + X_n$ is called the abstract Jordan decomposition of $X \in \mathfrak{g}$. Abstract, because as linear transformation, which $X \in \mathfrak{g} \subseteq \mathfrak{gl}(V)$ is, it already has a usual Jordan decomposition. Now Weyl's theorem states, that these two decompositions coincide!

Corollary: Let $\mathfrak{g}$ be a finite dimensional, semisimple Lie algebra, and (V, φ) a finite dimensional representation of $\mathfrak{g}$. If $X = X_s + X_n$ is the Jordan decomposition of $X \in \mathfrak{g}$, then

$$\varphi(X) = \varphi(X_s) + \varphi(X_n)$$

is the Jordan decomposition of (the matrix) $\varphi(X)$.

This might read a bit confusing for the first time. However, Weyl's theorem says, that we do not have to bother this confusion: there is only one Jordan decomposition, whether as given matrix of one of the classical Lie algebras, or as a vector within these Lie algebras where the decomposition is done along $\operatorname{ad}(X) \in \mathfrak{gl}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{gl}(V))$.

11 Weights.

Definition: Let (V, φ) be a finite dimensional representation of a nilpotent Lie algebra $\mathfrak{g}$. A linear function $\lambda \in \mathfrak{g}^*$ is called a **weight** of φ if there is a vector $0 \neq v \in V$ and an integer $m = m(v) \geq 1$ such that for all $X \in \mathfrak{g}$

$$(\varphi(X) - \lambda \cdot \operatorname{id}_{\mathfrak{g}})^m(v) = 0$$

In this case the set of all these vectors together with 0 form a linear subspace

$$V_\lambda = \{\, v \in V \mid (\varphi(X) - \lambda \cdot \operatorname{id}_{\mathfrak{g}})^m(v) = 0 \,\} \subseteq V$$

which is called **weight (sub)space** of φ corresponding to λ.

If $V_\lambda = V$ then φ is a nil representation called $\lambda-$**representation** and

$$\lambda(X) \cdot \dim V = \operatorname{tr}(\varphi(X))$$

Given two finite dimensional λ_i-representations (V_i, φ_i) of $\mathfrak{g}\,(i = 1, 2\,;\, \lambda_i \in \mathfrak{g}^*)$ then

$$(V_1 \otimes V_2, \varphi_1 \otimes \varphi_2) \text{ is a } (\lambda_1 + \lambda_2) - \text{representation of } \mathfrak{g}$$

Note that in case $\mathfrak{h}$ is a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$, $\mathfrak{h}$ is toral, thus diagonalizable, thus Abelian, thus a nilpotent Lie algebra, and the weight spaces corresponding to $(V, \varphi) = (\mathfrak{g}, \operatorname{ad}_{\mathfrak{h}})$ are the eigenspaces $E_\lambda(\mathfrak{h}) = V_\lambda$ and therefore precisely the root spaces. In this sense, weight spaces are the generalization of root spaces for arbitrary representations. The particular case of a Cartan subalgebra (with an arbitrary finite dimensional representation) is still a very important case, especially for the simple Lie algebras $\mathfrak{su}(n)$ which occur in particle and quantum physics.

So the general way to go for simple Lie algebras $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h}$ is: Consider the representation $(\mathfrak{g}, \mathrm{ad}_{\mathfrak{h}})$ in order to study the multiplicative structure of $\mathfrak{g}$ by **roots**, and in a second step consider arbitrary representations (V, φ) to study their actions on specific vector spaces by **weights**.

Theorem: Let be $\mathfrak{g}$ a nilpotent Lie algebra and (V, φ) a finite dimensional, complex representation of $\mathfrak{g}$. Then the weight subspaces of φ corresponding to distinct weights $\lambda_1, \ldots, \lambda_r$ are linearly independent and

$$V = \sum_{i=1}^{r} V_{\lambda_i} = \bigoplus_{i=1}^{r} V_{\lambda_i} \tag{17}$$

The sum is usually written with Σ although it is a direct sum.

12 Casimir elements.

In the previous parts we have seen that the Killing-form is a powerful tool to investigate semisimple Lie algebras. The Killing-form is the trace form of the adjoint representation. And as weights generalize roots, i.e. represent the step from the adjoint to arbitrary representations, we can also ask, how the Killing-form generalizes. Semisimple Lie algebras are direct sums of simple Lie algebras and their representations split accordingly. Therefore we may consider for the sake of simplicity a simple Lie algebra $\mathfrak{g}$ and a finite dimensional representation (V, φ). Since $\ker \varphi$ is an ideal of $\mathfrak{g}$, V is either a trivial $\mathfrak{g}-$module or φ is a faithful representation. Let us assume the latter and define the trace form

$$\beta(X, Y) := \mathrm{tr}(\varphi(X)\varphi(Y))$$

Then β is an associative, symmetric, nondegenerate, bilinear form on $\mathfrak{g}$ and for an ordered basis $\{ X_1, \ldots, X_n \}$ of $\mathfrak{g}$ there is a $\beta-$dual basis $\{ Y_1, \ldots, Y_n \}$ of $\mathfrak{g}$, i.e. $\beta(X_i, Y_j) = \delta_{ij}$.

$$c_\varphi = c_\varphi(\beta) := \sum_{i=1}^{n} \varphi(X_i)\varphi(Y_i)$$

is a linear transformation of V which commutes with $\varphi(\mathfrak{g})$; c_φ is called **Casimir element** of φ. We have $\mathrm{tr}\, c_\varphi = \dim(\mathfrak{g})$ and in case φ is irreducible, c_φ is a scalar multiplication with $c_\varphi = \dim(\mathfrak{g})/\dim(V)$.

13 Examples.

13.1 The Three Dimensional Simple Lie Algebra.

The (ordered) standard basis (X, H, Y) or sometimes (E, H, F) of the three dimensional simple Lie algebra $\mathfrak{sl}(2)$ is in terms of the Pauli matrices

$$\begin{aligned} X &= \frac{1}{2}\sigma_1 + \frac{1}{2}\,i\,\sigma_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ H &= \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ Y &= \frac{1}{2}\sigma_1 - \frac{1}{2}\,i\,\sigma_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned} \tag{18}$$

with the multiplications $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$ from which we get

$$\mathrm{ad}(\alpha, \beta, \gamma) = \mathrm{ad}(\alpha X + \beta H + \gamma Y) = \begin{bmatrix} -2\beta & -2\alpha & 0 \\ -\gamma & 0 & \alpha \\ 0 & 2\gamma & -2\beta \end{bmatrix}.$$

With the (irreducible) representation $1 = \mathrm{id} : \mathfrak{sl}(2) \subseteq \mathfrak{gl}(\mathbb{F}^2)$ we have a $\mathrm{id}\,-$dual basis $(Y, \frac{1}{2}H, X)$ and the Casimir element

$$c_{\mathrm{id}} = XY + \frac{1}{2}H^2 + YX = \frac{3}{2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{\dim(\mathfrak{sl}(2))}{\dim(\mathbb{F}^2)} \cdot \mathrm{id}_{\mathbb{F}^2}$$

The general classification of finite dimensional, irreducible, complex $\mathfrak{sl}(2, \mathbb{C})$ representations (V, φ) can be summarized as follows.

Theorem ($\mathfrak{sl}(2, \mathbb{C})$ **modules / representations**):

1. All weights λ, i.e. the eigenvalues of the semisimple (diagonizable) operation of H on V are integers and the weight spaces (eigenspaces) V_λ of this operation are one dimensional. The highest (maximal) weight be m and a vector $v_m \in V_m$ is called **maximal vector** or **vector of highest weight**.

2. $V = \bigoplus_{\substack{k=0 \\ \lambda=-m+2k}}^{m} V_\lambda = \bigoplus_{\substack{k=0 \\ \lambda=-m+2k}}^{m} \{v \in V \,:\, \varphi(H)(v) = \lambda \cdot v\}$

3. There is up to isomorphisms only one unique finite dimensional, irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$, resp. $\mathfrak{su}(2,\mathbb{C})$ per dimension of the representation space V.

4. Let v_m be a maximal vector. Then for $k=0,\ldots,m$ we define

$$v_{m-2k-2} := \frac{1}{(k+1)!}\,\varphi(Y)^{k+1}(v_m) \ \text{ and } \ v_{-m-2}=v_{m+2}=0$$

and get the following operation rules

$$\begin{aligned} \varphi(X)(v_{m-2k}) &= (m-k+1)\,v_{m-2k+2} \\ \varphi(H)(v_{m-2k}) &= (m-2k)\,v_{m-2k} \\ \varphi(Y)(v_{m-2k}) &= (k+1)\,v_{m-2k-2} \end{aligned}$$

5. If (V,φ) is any (not necessarily irreducible) finite dimensional representation, then the eigenvalues are all integers, and each occurs along with its negative an equal number of times. In any decomposition of V into irreducible submodules, the number of summands is precisely $\dim V_0 + \dim V_1$.

13.2 The Adjoint Representation.

A representation consists actually of three parts: What is represented, as what is it represented, and how is it represented? Thus it makes a big difference whether we talk about a representation **of a Lie algebra** or a representation **on a Lie algebra**. In case of the adjoint representation, we have both with the same name:

The adjoint representation of a Lie group G **on its Lie algebra** by conjugation:

$$\begin{aligned} \mathrm{Ad}\,:\, G &\longrightarrow GL(\mathfrak{g}) \\ g &\longmapsto \left(X \longmapsto gXg^{-1}\right) \end{aligned}$$

and the adjoint representation **of a Lie algebra** $\mathfrak{g}$ on itself by left (Lie) multiplication:

$$\begin{aligned} \mathrm{ad}\,:\, \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g})) \\ X &\longmapsto (Y \longmapsto [X,Y]) \end{aligned}$$

Both adjoint representations are connected by the formula ($X\in\mathfrak{g}$)

$$\mathrm{Ad}(\exp X) = \exp(\mathrm{ad}(X)) \tag{19}$$

This formula can be visualized by the commutativity of the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & GL(\mathfrak{g}) \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathfrak{gl}(\mathfrak{g}) \end{array}$$

between Lie groups (analytic manifolds in which group multiplication and inversion are analytical functions) in the top row and their tangent spaces at $g = 1$ (Lie algebras) in the bottom row. It reflects an integration process, similar to the standard ansatz when solving differential equations by assuming an exponential function as solution. In this sense the adjoint representation of the Lie algebra is the differential of the adjoint representation of the Lie group, and the adjoint representation of the Lie group the integrated adjoint representation of the Lie algebra. It integrates $0 \in \mathfrak{g}$ to $1 \in G$, resp. the tangent space at $g = 1$ to the connection component of the group identity. The differentiation process can be achieved by considering flows on the manifolds (cp. [6] or [12],[13]).

It can be proven, that given an analytic group homomorphism $\varphi : G_1 \longrightarrow G_2$ between two Lie groups with the differential $D\varphi$

$$\mathrm{Ad}(\varphi(g)) \circ D\varphi = D\varphi \circ \mathrm{Ad}(g) \quad (g \in G_1)\,. \tag{20}$$

Linear transformations generally do not commute, so that the fundamental formula of the exponential function $e^{a+b} = e^a \cdot e^b$ does not apply here. Of course we still have $\exp(c \cdot X) = e^c \cdot \exp(X)$ for the scalar multiplication, but it is also of interest to know, how the product of two exponentiated Lie algebra vectors behave with respect to other Lie algebra vectors.

Theorem (Baker-Campbell-Hausdorff Formula):

$$\begin{aligned}\exp(X) \cdot \exp(Y) = \exp\Bigg(& X + Y + \frac{1}{2}[X, Y] + \\ & + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] \\ & - \frac{1}{24}[Y, [X, [X, Y]]] \\ & - \frac{1}{720}([[[[X, Y], Y], Y], Y] + [[[[Y, X], X], X], X]) \\ & + \frac{1}{360}([[[[X, Y], Y], Y], X] + [[[[Y, X], X], X], Y]) \\ & + \frac{1}{120}([[[[Y, X], Y], X], Y] + [[[[X, Y], X], Y], X]) + \cdots \Bigg)\end{aligned}$$

13.3 The Natural Representation of a Linear Lie Algebra.

A linear Lie algebra is a subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ of linear transformations on a vector space V. Thus there is a natural representation (V, id) given by

$$\mathrm{id}_{\mathfrak{g}}(X)(v) = X.v = X \cdot v = X(v)$$

There is a subtlety with the definition here. The natural representation of a linear Lie algebra is only given if the multiplication is defined by its subalgebra property as

$$[X, Y](v) = X(Y(v)) - Y(X(v))$$

which is normally not especially mentioned. But theoretically it would be possible, that the Lie multiplication is defined differently, in which case this has to be mentioned. E.g. we could define another, Abelian multiplication on $\mathfrak{sl}(2, \mathbb{R}) \subseteq \mathfrak{gl}(\mathbb{R}^2)$ by just setting $[X, Y] = 0$. In this case we have a three dimensional Euclidean space as Lie algebra, which then should be written as $\mathbb{R}^3$ instead. It can also happen, that a definition doesn't immediately show, that the Lie algebra is isomorphic to a certain linear one. Remember Ado's theorem that all (real or complex, finite dimensional) Lie algebras are isomorphic to a linear one. The natural representation is therefore an important representation, not the least because all results from linear algebra immediately apply. Note that a linear Lie group $G \subseteq GL(V)$ and its Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ operate on, resp. are represented as linear transformations of the same vector space V.

13.4 The Algorithm Manifold.

Lie multiplication, even if not defined as subsequent application of a linear transformation or other operators is still a bilinear transformation $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ and as such can be written as

$$\beta(X, Y) = [X, Y] = \sum_{i}^{r} u_i(X) \cdot v_i(Y) \cdot W_i$$

with $(1, 2)$ tensors $u_i \otimes v_i \otimes W_i \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ which is called a bilinear algorithm. The set of all bilinear algorithms of β builds an affine variety which is called **algorithm manifold** of β. The group

$$\Gamma(\beta) = \{ \varphi^* \otimes \psi^* \otimes \chi^* \in GL(\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}) \, : \, [X, Y] = \chi([\varphi(X), \psi(Y)]) \}$$

is called **isotropy group** of β and is an example of a group operation on $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$. The Lie algebras here serve as representation space and the group elements are those which leave the Lie multiplication invariant, esp. its **tensor rank** r. With the embedding $\alpha^{-1} \longmapsto \alpha^* \otimes \alpha^* \otimes \alpha^{-1}$ we have group monomorphism from $\mathrm{Aut}(\mathfrak{g}) \longrightarrow \Gamma(\mathfrak{g})$. It can be shown that for simple Lie algebras (as well as for their Borel subalgebras) the automorphisms are the only elements of the isotropy group with the exception that

$$\Gamma(\mathfrak{sl}(2,\mathbb{F})) \cong GL(SL(2,\mathbb{F}))/\mathbb{F}^*$$

This means an exception for the tangent space of the unitary group $SU(2)$, too, since $\mathfrak{su}(2) \cong \mathfrak{sl}(2)$. In other words: Due to its minimality (cp. its Dynkin diagram "$\circ$") the three dimensional simple Lie algebra behaves a little bit different than other simple Lie algebras.

13.5 A Related Lie Algebra as Representation.

The investigation of the isotropy group leads to the consideration of transposable transformations for which $[\tau(X), Y] = [X, \tau^\dagger(Y)]$, and with the standard split $\tau = \frac{1}{2}(\tau + \tau^\dagger) + \frac{1}{2}(\tau - \tau^\dagger)$ into a symmetric $[(\tau + \tau^\dagger)(X), Y] = [X, (\tau + \tau^\dagger)(Y)]$ and an antisymmetric part $(\tau - \tau^\dagger)$ to

$$A(\mathfrak{g}) = \{\, \alpha \in \mathfrak{gl}(\mathfrak{g}) \,:\, [\alpha(X), Y] = -[X, \alpha(Y)] \text{ for all } X \in \mathfrak{g} \,\}$$

which turns out to be a Lie algebra again, the Lie algebra of **antisymmetric transformations** of $\mathfrak{g}$. As always with such definitions, the question of existence has to be answered, or more precisely, whether $A(\mathfrak{g})$ can be different from the zero Lie algebra. The trivial case is of course when $\mathfrak{g}$ is Abelian, in which case $A(\mathfrak{g}) = \mathfrak{gl}(\mathfrak{g})$. On the other hand, it can be shown that indeed $A(\mathfrak{g}) = \{\,0\,\}$ whenever $\mathfrak{g}$ is a simple Lie algebra. However $\mathfrak{gl}(\mathfrak{g}) \supsetneq A(\mathfrak{g}) \neq \{\,0\,\}$ for any solvable, non Abelian Lie algebra as e.g. the Borel subalgebras of simple Lie algebras:

Let $\mathfrak{g}$ be a finite dimensional complex, non Abelian Lie algebra. By Lie's theorem there is a one dimensional ideal $\mathfrak{I} = \langle I \rangle \subseteq \mathfrak{g}$, hence $[X, I] = \lambda(X)I$ for some $\mathfrak{g}^* \ni \lambda \neq 0$. With $\alpha(X) := \lambda(X)I$ we get a non trivial antisymmetric transformation.

In a way the antisymmetric Lie algebra $A(\mathfrak{g})$ measures the point where $\mathfrak{g}$ lies between simple (most structured) and Abelian (least structured) Lie algebras. The transformation defined above is by the way the only one for Borel subalgebras of simple Lie algebras (with the exception of $\mathfrak{sl}(2)$). So the less structure $\mathfrak{g}$ has, the more structure has $A(\mathfrak{g})$ and vice versa.

Theorem (Antisymmetric Transformations): $A(\mathfrak{g})$ is a $\mathfrak{g}-$module, i.e.

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{A}(\mathfrak{g})) \\ X &\longmapsto (\alpha \longmapsto [\mathrm{ad}(X), \alpha] = \mathrm{ad}(X) \circ \alpha - \alpha \circ \mathrm{ad}(X)) \end{aligned} \tag{21}$$

defines a representation of $\mathfrak{g}$ on $A(\mathfrak{g})$.

This follows from repeated applications of the Jacobi identity and the definition of an antisymmetric transformation. Since $A(\mathfrak{g})$ is again a Lie algebra, we can ask for $A(A(\mathfrak{g})), A(A(A(\mathfrak{g})))$ etc. or build the semidirect product $\mathfrak{g} \ltimes A(\mathfrak{g})$ and then repeat the process. Even

$$[X, Y] \longmapsto \alpha([X, Y])$$

with a fixed antisymmetric transformation $\alpha \in A(\mathfrak{g})$ defines again a Lie algebra structure on the same vector space $\mathfrak{g}$. In this sense, the antisymmetric transformations build a large pool of possible representations.

13.6 Differential Operators.

Lie algebras and differential operators are closely related in the sense that a set of differential operators can build the basis for a Lie algebra which operates on some Hilbert space, i.e. in general infinite dimensional representation spaces.

E.g. we define $D_n := x^n \cdot \dfrac{d}{dx} \quad (n \in \mathbb{Z})$, then

$$[D_n, D_m] = (m - n)D_{n+m-1}$$

The Lie algebra generated by these differential operators is in general infinite dimensional and operates on the Hilbert space of smooth real functions $C^\infty(\mathbb{R})$. We get a finite dimensional example with

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{gl}(C^\infty(\mathbb{R})) \\ \mathfrak{g} &:= \langle D_{-n+1}, D_1, D_{n+1} \rangle \\ & D_{-n+1}(f) = x^{-n+1} f\,', \; D_1(f) = x f\,', \; D_{n+1} = x^{n+1} f\,' \\ & [D_{-n+1}, D_1] = nD_{-n+1}\,, \; [D_{-n+1}, D_{n+1}] = 2nD_1\,, \; [D_1, D_{n+1}] = nD_{n+1} \end{aligned}$$

which is the three dimensional simple Lie algebra, an isomorphic copy of $\mathfrak{sl}(2, \mathbb{R})$.

Another example which also operates on $V = C^\infty(\mathbb{R})$ is given by

$$X_0 = \frac{1}{2}\left(\frac{d^2}{dx^2} + x^2 \cdot \mathrm{id}_V\right) \, , \, X_1 = \frac{d}{dx} \, , \, X_2 = x \cdot \mathrm{id}_V \, , \, X_3 = \mathrm{id}_V$$

which yields the non zero multiplications

$$[X_1, X_2] = X_3 \, , \, [X_0, X_1] = -X_2 \, , \, [X_0, X_2] = X_1$$

It is a four dimensional, solvable, real Lie algebra called **oscillator algebra**. It has a central element X_3 and with $\langle X_1, X_2, X_3 \rangle$ a copy of the Heisenberg algebra as nilradical $\mathfrak{H} = [\mathfrak{g}, \mathfrak{g}]$, and $\mathfrak{h} = \langle X_0, X_3 \rangle$ as Cartan subalgebra:

$$\mathfrak{h} \supsetneq \mathfrak{g} = \mathfrak{H} \rtimes \mathbb{R} \cdot X_0 \, .$$

14 Epilogue

I hope I could have shown how rich and complex the world of Lie algebra representations is. For further investigations in this area I recommend the sources [18], [19], [20] and the literature quoted therein. Other interesting key words to search for are: *quasi-exact solvability, Schrödinger operator, oscillator algebra, realization of the Lie algebra, Lie algebra of differential operators, highest weights.*

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